Cal Poly

Algebra Qualifying Exam

January 15, 2022

Directions.

- **Show all work!** Write everything down. Insufficient justification can mean no credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators and phones.

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Groups

1. Let $\sigma : G \longrightarrow H$ be a group epimorphism. Let N be a normal subgroup of G and $K = \sigma(N)$, the image of N in H.

(a) Prove that K is a normal subgroup of H. Give an example to show that this is not true if σ is not onto.

(b) Under what conditions does σ induce a homomorphism $G/N \longrightarrow H/K$, and when is this an isomorphism? Prove your answer.

2. The dihedral group D_4 is the group of 8 rigid motions of a square. Prove that D_4 is not the internal direct product of two of its proper subgroups.

Rings

3. Let A be a commutative ring with 1. The *dimension* of A is the maximum length d of a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$. Prove that if A is a PID, the dimension of A is at most 1.

4. Let $\mathbb{Z}_2 = \{0, 1\}$ be the field of 2 elements. The quotient ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field of cardinality 8, containing $\mathbb{Z}_2$. Let $\pi : \mathbb{Z}_2[x] \longrightarrow \mathbb{Z}_2[x]/(x^3 + x + 1)$ be the natural projection.

(a) Write down a set of 8 distinct coset representatives for the elements of this field.

(b) Determine the multiplicative inverse of $\pi(x)$ in terms of your coset representatives.

Vector Spaces

5. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the orthogonal projection to a 1-dimensional linear subspace $L \subset \mathbb{R}^3$.

(a) List the eigenvalues of T.

(b) Write the characteristic polynomial $p_T(x)$ for T.

(c) Is T diagonalizable? Justify your answer.