Cal Poly

Algebra Qualifying Exam

June 4, 2022

- **Show all work!** Write everything down. Insufficient justification can mean no credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators and phones.

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Groups

1. (5 points) Let N be a finite normal subgroup of G. Prove there is a normal subgroup M of G such that $[G : M]$ is finite and $nm = mn$ for all $n \in N$ and $m \in M$. [*Hint*: You may use the fact that the *centralizer* $C(h) := \{g \in G : ghg^{-1} = h\}$ is a subgroup G.]

2. (5 points) Let S_7 denote the symmetric group.

(a) Give an example of two nonconjugate elements of S_7 that have the same order.

(b) If $g \in S_7$ has maximal order, what is the order of g?

(c) Does the element g that you found in part (b) lie in A_7? Fully justify your answer.

(d) Determine whether the set $\{h \in S_7 : |h| = |g|\}$ is a single conjugacy class in S_7, where g is the element found in part (b).

Rings

3. (5 points) Let R be a commutative ring with 1. Use theorems in ring theory to prove:

(a) (x) is a prime ideal in $R[x]$ if and only if R is an integral domain.

(b) (x) is a maximal ideal in $R[x]$ if and only if R is a field.

4. (5 points) Let R be a commutative ring with 1, and $\sigma : R \longrightarrow R$ a ring automorphism.

(a) Show that $F = \{r \in R : \sigma(r) = r\}$ is a subring of R (with 1).

(b) Show that if σ^2 is the identity map on R, then each element of R is the root of a monic polynomial of degree 2 in $F[x]$, where F is as in (a).

Vector Spaces

5. (5 points) Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

(a) Compute the characteristic polynomial $p_A(x)$ of A. It has integer roots.

(b) For each eigenvalue λ of A, find a basis for the eigenspace E_λ.

(c) Determine if A is diagonalizable. If so, give matrices P and B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully why A is not diagonalizable.