Cal Poly

Algebra Qualifying Exam

June 5, 2021

Directions.

- Do not leave Zoom until your exam is uploaded. If you get disconnected, scan your exam immediately (with time signature).
- **Show all work!** Write everything down. No justification, no credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators, phones, notes, text, internet, and communication with other people.

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1. Prove from the definition alone that there are no nonabelian groups of order less than 5.

2. Let A_5 denote the alternating group on a 5-element set $\{1,2,3,4,5\}$. The set of automorphisms of A_5 form a group, denoted $\mathrm{Aut}(A_5)$. The group of *conjugations* of A_5, denoted $\mathrm{Conj}(A_5)$, is the subgroup of $\mathrm{Aut}(A_5)$ consisting of automorphisms of the form $\gamma_s := s\,(-)\,s^{-1}$ where $s \in A_5$. Explicitly, $\gamma_s(x) = sxs^{-1}$ for any $x \in A_5$.

(a) Prove that the function $\gamma : A_5 \longrightarrow \mathrm{Conj}(A_5)$, taking $s \in A_5$ to γ_s, is a surjective homomorphism.

(b) Prove that A_5 is isomorphic to $\mathrm{Conj}(A_5)$.

3. Let $\mathbb{Z}[X]$ be the ring of polynomials with integer coefficients, and let $K \subset \mathbb{Z}[X]$ be the kernel of the "evaluation at 1" homomorphism

$$\begin{aligned}\varepsilon_1 : \mathbb{Z}[X] &\longrightarrow \mathbb{Z}_3\\ f(X) &\longmapsto [f(1)]_3\end{aligned}$$

(a) Characterize K as a set.

(b) Determine whether K is a maximal ideal. Fully justify your conclusion.

(c) Determine whether K is a principal ideal. Justify by either exhibiting a generator or proving that there isn't one.

4. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Determine whether A is diagonalizable, and if so, give its diagonal form along with a diagonalizing matrix.

(b) Compute A^{42}. Remember to show all work.

5. Let $\mathbb{F}_9$ denote the field of 9 elements.

(a) Show that each nonzero $a \in \mathbb{F}_9$ is a root of $X^8 - 1 = (X-1)(X+1)(X^2+1)(X^4+1) \in \mathbb{F}_3[X]$.

(b) Use the pigeonhole principle to prove that $\mathbb{F}_9$ has an element of multiplicative order 8. (Include a proof that the pigeonhole principle applies.)