Real Analysis Qualifying Exam, June 5, 2022

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Good luck!

1. Prove that every convergent sequence of real numbers has a maximum or a minimum value.

2. Let (f_n) be a sequence of increasing functions on $[a, b]$ with $\sum_{n=1}^{\infty} f_n(x)$ absolutely convergent when $x = a$ and when $x = b$. Show that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for each x in $[a, b]$ and also that the series converges uniformly on $[a, b]$.

3. Find, with proof, the maximum number of real roots of the function $f(x) = x^{16} + ax + b$, where a and b are real numbers.

4. (a) State a definition for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

(b) Let

$$f(x) = \begin{cases} 1, & 1 \leq x < 2 \\ 10, & x = 2 \\ 2, & 2 < x \leq 3. \end{cases}$$

Prove, using your definition, that f is integrable on $[1, 3]$.

5. Suppose $f : R \to R$ is a contraction, i.e., there is a number $0 < k < 1$ such that for all x and y, $|f(x) - f(y)| \leq k\,|x - y|$. Fix a number x_0 and define $x_n = f(x_{n-1})$ for each $n \geq 1$. Prove the sequence (x_n) is Cauchy.