Analysis Qualifying Exam Problem Bank

Introduction

This list is comprised of potential problems for the qualifying exam in analysis. Problems marked with a **(*)** indicate questions that may be modified in some way while retaining the same basic structure. For example, Problem 5 **(*)** could be asked of a different sequence. Other notable types of problems likely to be marked with a **(*)** include M-test questions on uniform convergence and continuity of series of functions and problems involving showing that functions are integrable from the definition of Riemann integration.

Sequences

Appeared on: S14

Problem 1.(a) Assume that $\sum_{n=1}^{\infty} a_n$ is convergent and $\{b_n\}$ is a bounded sequence. Does $\sum_{n=1}^{\infty} a_n b_n$ converge? Prove or provide a counterexample.

(b) Assume that $\sum_{n=1}^{\infty} |a_n|$ is convergent and $\{b_n\}$ is a bounded sequence. Does $\sum_{n=1}^{\infty} a_n b_n$ converge? Prove or provide a counterexample.

Appeared on: S15

Problem 2. Show that the least upper bound property of the real numbers implies the Cauchy completeness property; that is, show that the property that every bounded set of real numbers has a least upper bound implies that every Cauchy sequence of real numbers converges in $\mathbf{R}$.

Appeared on: F14

Problem 3. Let (x_n) be a sequence in $\mathbb{R}$ with $|x_n - x_{n+1}| < \frac{1}{n}$, for all $n \in \mathbb{N}$.

(a) If (x_n) is bounded, must (x_n) converge?

(b) If the subsequence (x_{2n}) converges, must (x_n) converge?

Appeared on: W21

Problem 4. Suppose that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $\mathbb{R}$. Prove, using the definition of a Cauchy sequence, that $\{|a_n - b_n|\}$ converges in $\mathbb{R}$.

Appeared on: F20

Problem 5. **(*)** Let $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + s_n}$ for $n = 1, 2, 3, \ldots$.

(a) Show that $s_n \leq 2$ for all n.

(b) Show that $\{s_n\}$ converges and then compute the limit of the sequence.

Appeared on: W20

Problem 6. **(*)** Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$

(a) Prove that $\{a_n\}$ is increasing.

(b) Prove that $\{a_n\}$ converges.

Appeared on: F17

Problem 7. Prove that the sequence $\{a_n\}$, where

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}},$$

converges and compute its limit.

Appeared on: F16

Problem 8. **(*)**

1. Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \dfrac{1}{4 - x_n}$ converges.
2. Explicitly compute the limit of the sequence in part (a).

Appeared on: S16

Problem 9.(a) Argue from the definition of Cauchy sequence that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is $\{a_n b_n\}$.

(b) Give an example of a sequence $\{a_n\}$ with $\lim |a_{n+1} - a_n| = 0$ but which is *not* Cauchy.

Appeared on: F15

Problem 10. **(*)** Let $\{x_n\}$ be a sequence of real numbers satisfying

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|,$$

for all $n \geq 1$, where $0 < C < 1$ is a constant. Prove that $\{x_n\}$ converges.

Appeared on: F24

Problem 11. 1. Exhibit, with proof, a sequence of real numbers which has $[0, 1]$ as its set of limit points.

2. Does there exist a sequence with $(0, 1)$ as its set of limit points? Give an example with proof or prove that no such sequence exists.

Appeared on: W23

Problem 12. **(*)** Show that the sequence (x_n) is Cauchy, where

$$x_n = \int_1^n \frac{\cos t}{t^2}\, dt.$$

Appeared on: S22

Problem 13. Prove that every convergent sequence of real numbers has a maximum or minimum value.

Appeared on: S22

Problem 14. **(*)** Suppose that for a function $f : \mathbf{R} \to \mathbf{R}$, there is a number $k \in (0,1)$ such that for all $x, y \in \mathbf{R}$,

$$|f(x) - f(y)| \leq k\,|x - y|\,.$$

Fix a number x_0, and define a sequence by

$$x_n = f(x_{n-1})$$

for each $n \geq 1$. Prove that (x_n) is a Cauchy sequence.

Appeared on: W22

Problem 15. Let (x_n) be a sequence such that (x_{2n}), (x_{2n+1}) and (x_{3n}) are convergent. Show that (x_n) is convergent.

Series

Appeared on: S18

Problem 16. **(*)** Prove that the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges by showing that the sequence of partial sums is Cauchy.

Appeared on: W22

Problem 17. Suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series of positive terms. Show that $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}$ are also convergent.

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Problem 18. Solve the following:

(a) Suppose that $(f_n) \to f$ uniformly and $(g_n) \to g$ uniformly on a subset A of $\mathbb{R}$. Prove that if f and g are bounded on A, then $(f_n g_n) \to fg$ uniformly on A.

(b) Show that (a) may be false if g is unbounded. *Hint:* Consider $f_n(x) = 1/n$ and $g_n(x) = x + 1/n$. Prove that the convergence $(f_n g_n) \to fg$ in this case is not uniform on $\mathbb{R}$.

Continuity

Problem 19. Suppose that $f : [0, \infty) \to \mathbf{R}$ is a continuous, increasing, bounded function. Prove that f is uniformly continuous on $[0, \infty)$.

Appeared on: W25

Problem 20. Let $\{f_n\}$ be a sequence of functions defined on $A \subseteq \mathbb{R}$.

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(a) Prove if each f_n is uniformly continuous on A and (f_n) converges uniformly on A to a function f, then f is uniformly continuous on A.

(b) Give a counter example to show that (a) is false if we assume pointwise convergence instead of uniform convergence.

Problem 21.(a) Let $f : \mathbf{R} \to \mathbf{R}$ be uniformly continuous. Show that if $\{x_n\} \subset \mathbf{R}$ is a Cauchy sequence of real numbers, then $\{f(x_n)\}$ is a Cauchy sequence.

Appeared on: S14

(b) Suppose that f_n is a sequence of continuous functions that converge uniformly on a subset $A \subset \mathbf{R}$ to a function f. Show that f is continuous on A.

Problem 22. **(*)** Consider the function

Appeared on: W21

$$g(x) = \begin{cases} e^x, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

Find, with proof, the set $C = \{x \in \mathbb{R} \mid g \text{ is continuous at } x\}$.

Problem 23. **(*)** Define $f : (-1, 0) \cup (0, 1) \to \mathbb{R}$ by

Appeared on: F20

$$f(x) = \begin{cases} 4, & x \in (-1, 0) \\ 5, & x \in (0, 1) \end{cases}.$$

1. Show that f is continuous on $(-1, 0) \cup (0, 1)$.
2. Show that f is not uniformly continuous on $(-1, 0) \cup (0, 1)$.

Problem 24. Let (f_n) be a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose f_n is bounded for each $n \in \mathbb{N}$.

Appeared on: S20

(a) Prove that if $f_n \to f$ uniformly on $\mathbb{R}$, then f is bounded.

(b) If each f_n is continuous and $f_n \to f$ pointwise on $\mathbb{R}$, does f have to be bounded? Give a proof or a counterexample.

Appeared on: F19

Problem 25. **(*)** Show that the sequence of functions

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$$

converges pointwise to $f(x) = 0$ on $[0, 1]$, but does not converge uniformly.

Appeared on: F18

Problem 26. **(*)**

(a) Define what it means for $f : A \subseteq \mathbb{R} \to \mathbb{R}$ to be uniformly continuous.

(b) Use the definition to show that $f(x) = 1/x$ is uniformly continuous on $(1, 2)$.

(c) Show that $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

Appeared on: S18

Problem 27.(a) Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. Prove that if each f_n is continuous at $c \in A$, then f is continuous at c.

(b) Give an example to show that the result above is false if we only assume that (f_n) converges pointwise to f on A.

Appeared on: F16

Problem 28. **(*)** Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{\sin(nx)}{1 + nx}.$$

(a) Show that f_n converges pointwise on $[0, \infty)$ and find the pointwise limit f.

(b) Show that $f_n \to f$ uniformly on $[a, \infty)$ for every $a > 0$.

(c) Show that f_n does not converge uniformly to f on $[0, \infty)$.

Appeared on: S16

Problem 29. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions that converges uniformly on $\mathbb{R}$ to a function f. Let $\{x_n\}$ be a sequence of real numbers that converges to $x_o \in \mathbb{R}$. Prove that $\{f_n(x_n)\} \to f(x_o)$.

Appeared on: F15

Problem 30. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty.$$

Prove that f attains an absolute minimum value on $\mathbb{R}$. In other words, prove that there exists a real number c such that $f(c) \leq f(x)$ for all $x \in \mathbb{R}$.

Appeared on: W25

Problem 31. Suppose that $f : [0, \infty) \to \mathbf{R}$ is a continuous, increasing, bounded function. Prove that f is uniformly continuous on $[0, \infty)$.

Appeared on: W24

Problem 32. A zero of a continuous function is called *isolated* if there exists an open set containing that zero but no other zeros of f.

1. Given an example of a continuous functions $f : (0, 1) \to \mathbf{R}$ with infinitely many isolated zeros.
2. If $f : [0, 1] \to \mathbf{R}$ is continuous and all of its zeros are isolated, show that f has only finitely many zeros on $[0, 1]$.

Appeared on: F24

Problem 33. (*)

1. Give a definition for a function $f : [a, b] \to \mathbf{R}$ to be uniformly continuous.
2. Using your definition (and not a theorem) prove that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 2]$.

Appeared on: F23

Problem 34. Suppose that $f(x)$ is continuous and unbounded on $[a, b)$. Prove that $\lim_{x \to b^-} f(x)$ does not exist.

Appeared on: S23

Problem 35. (*) For each $n = 1, 2, 3, \ldots$, the function

$$f_n(x) = \frac{nx}{e^{nx}}$$

is continuous on $[0, 2]$. Find the pointwise limit function $f(x) = \lim_{n \to \infty} f_n(x)$ and show that (f_n) does not converge uniformly to f.

Appeared on: S23

Problem 36. Let f be continuous on $[0, 1]$ with $f(x) > 0$ for all $x \in [0, 1]$. Let $S = \sup\{f(x) : x \in [0, 1]\}$. Show that for every $\varepsilon > 0$, there is some open interval I on which $f(x) > S - \varepsilon$.

Appeared on: W23

Problem 37. Show that if $f_n(x)$ is a uniformly continuous function on $[0, 1]$ for each $n = 1, 2, 3, \ldots$ and $f_n \to f$ uniformly on $[0, 1]$, then $f(x)$ is also uniformly continuous on $[0, 1]$.

Appeared on: S24

Problem 38. (*) Define a sequence of functions by

$$f_n(x) = \frac{nx^n}{1 + nx^n}$$

for $n = 1, 2, 3, \ldots$.

(a) Find the pointwise limit $f(x)$ for each $x \in [0, \infty)$.

(b) Prove (f_n) does not converge uniformly on $[0, \infty)$.

(c) Prove (f_n) converges uniformly on $[1,2]$.

Appeared on: W22

Problem 39. **(*)** Consider the sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$ given by

$$f_n(x) = \frac{nx}{\sqrt{1+n^2x^2}}.$$

Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$. Does (f_n) converge to f uniformly on $\mathbf{R}$? Justify your answer.

Derivatives and the Mean Value Theorem

Problem 40. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with a continuous derivative. Suppose there exist four distinct points w, x, y, z in $\mathbb{R}$ with $f(w) = f(x)$ and $f(y) = y$ and $f(z) = z$. Prove that there is a point u where $f'(u) = \frac{1}{2}$.

Appeared on: S21

Problem 41. **(*)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that f is differentiable, that $f(0) = 1$, and that $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

Appeared on: W21

Problem 42. **(*)** Prove that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ and $f'(x) \geq 1$ for all $x \neq 0$. [*Hint:* Use the Mean Value Theorem.]

Appeared on: F20

Problem 43. A function $f : \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous* on a set $A \subseteq \mathbb{R}$ if there exists a constant $M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$.

Appeared on: S20

(a) Assume that f is a differentiable function on $\mathbb{R}$ and that f' is continuous on $[a, b]$. Prove that f is Lipschitz on $[a, b]$.

(b) Prove that a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Problem 44. **(*)** A function $f : \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous* on a set $A \subseteq \mathbb{R}$ if there exists a constant $M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$.

Appeared on: S15

(a) Show that $f(x) = \sqrt{x}$ is Lipschitz continuous on $[1, \infty)$ but not $[0, \infty)$.

(b) Prove that a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Problem 45. **(*)** Show that the function

Appeared on: S20

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is differentiable only at $x = 0$.

Problem 46. **(*)**

Appeared on: F19

(a) State the Mean Value Theorem.

(b) Use the Mean Value Theorem to prove that $|\tan x| \geq |x|$ for all $x \in (-\pi/2, \pi/2)$.

Appeared on: W19

Problem 47. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq (x-y)^2, \quad \text{for all } x, y \in \mathbb{R}.$$

Show that f is a constant function on $\mathbb{R}$. (*Hint*: Is f differentiable?)

Appeared on: W19

Problem 48.(a) Suppose that f is a real valued function on $(0, \infty)$ whose derivative exists and is bounded on $(0, \infty)$. Prove that f is uniformly continuous on $(0, \infty)$.

(b) Give an example of a differentiable real valued function f on $(0, \infty)$ that is uniformly continuous on $(0, \infty)$ yet f' is unbounded on $(0, \infty)$.

Appeared on: F18

Problem 49. **(*)** Suppose that f is differentiable on $\mathbb{R}$ and that $f'(x) \leq 4$ for all $x \in \mathbb{R}$. Prove that there is at most one point $x > 2$ such that $f(x) = x^2$.

Appeared on: S18

Problem 50. **(*)** Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $|f'(x)| < 1$ for all $x \in \mathbb{R}$.

(a) f has a fixed point at x_0 if $f(x_0) = x_0$. Prove that f has at most one fixed point.

(b) Show that the following function satisfies $|f'(x)| < 1$ for all $x \in \mathbb{R}$ but has no fixed points:

$$f(x) = \ln(1 + e^x)$$

Appeared on: F16

Problem 51.(a) Prove that $\ln x \leq x - 1$, for all $x > 0$.

(b) Prove that $\ln x \geq x - 1 - \frac{1}{2}(x-1)^2$, for all $x \geq 1$, and that $\ln x \leq x - 1 - \frac{1}{2}(x-1)^2$, for all $0 < x \leq 1$.

Appeared on: S16

Problem 52. Let f be a function that is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Show that if $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in (0, 1)$, then $f(x) = 0$ for all $x \in [0, 1]$.

Appeared on: F15

Problem 53. **(*)** Prove that for all real numbers x and y,

$$|\cos^2(x) - \cos^2(y)| \leq |x - y|.$$

Appeared on: S24

Problem 54. (*) Suppose that f is continuous on $[0,1]$. Show that there is some $c \in [0,1]$ with

$$\int_0^1 x^2 f(x)\,dx = \frac{1}{3} f(c).$$

Appeared on: F24

Problem 55. (*)

1. State the Mean Value Theorem.
2. Show that if $f : \mathbf{R} \to \mathbf{R}$ is differentiable, $f(0) = 0$, and for all x, $|f'(x)| < |x|^3$, then $|f(x)| \le x^4$ for all x.

Appeared on: W24

Problem 56. Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable everywhere on (a,b) except perhaps at one number $c \in (a,b)$, and let $\lim_{x\to c} f'(x)$ exist. Show that f is differentiable at c and $f'(c) = \lim_{x\to c} f'(x)$.

Appeared on: F23

Problem 57. Let $f : [a,b] \to \mathbf{R}$ be continuous and twice differentiable on (a,b). Assume that the line segment from $A = (a, f(a))$ to $B = (b, f(b))$ intersects the graph of f in a third point different from A and B. Show that $f''(c) = 0$ for some $c \in (a,b)$.

Appeared on: S23

Problem 58. Prove that if f is a function which is differentiable on all of $\mathbf{R}$ and $f'(x) > 0$ for all x, then f is injective.

Appeared on: W23

Problem 59. Suppose f and g are continuous on $[a,b]$ and f' and g' are continuous on (a,b), with $f(a) = g(a)$ and $f(b) = g(b)$. Prove there is a number $c \in (a,b)$ such that the line tangent to the graph of f at the point $(c, f(c))$ is parallel to the line tangent to the graph of g at $(c, g(c))$.

Appeared on: S22

Problem 60. (*) Find, with proof, the maximum number of real roots of the function $f(x) = x^{16} + ax + b$ where a and b are real numbers.

Appeared on: W22

Problem 61. (*) A function $f : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous if there is a constant $M \ge 0$ such that

$$|f(x) - f(y)| \le M\,|x - y|$$

for all $x, y \in \mathbf{R}$.

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable and $f' : \mathbf{R} \to \mathbf{R}$ is bounded. Prove that f is Lipschitz continuous.
2. Give an example, with proof, of a function $f : \mathbf{R} \to \mathbf{R}$ that is differentiable but not Lipschitz continuous.
3. Give an example, with proof, of a function $f : \mathbf{R} \to \mathbf{R}$ that is Lipschitz continuous but not differentiable.

Series of functions

Problem 62. **(*)** Let $a > 0$. For each $n \in \mathbb{N}$, consider the function $f_n : \mathbf{R} \to \mathbf{R}$ given by $f_n(x) = \dfrac{\sin(x/n)}{\sqrt{1+n^2}}$.

Appeared on: S15

(a) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[-a, a]$.

(b) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ is continuously differentiable on $(-a, a)$.

Appeared on: S21

Problem 63. Suppose f is continuous on $[0, 1]$ and $|f(x)| < 1$ for all x on $[0, 1]$. Prove that F is uniformly continuous on $[0, 1]$, where

$$F(x) = \sum_{k=1}^{\infty} (f(x))^k.$$

Appeared on: W21

Problem 64. **(*)** Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

(a) Find the domain of $f(x)$ precisely.

(b) Prove that f is uniformly continuous on this domain.

Appeared on: W20

Problem 65. **(*)** Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(x^n)}{n^2 x^n}.$$

(a) Prove that f is continuous on $[1, \infty)$.

(b) Prove that, in fact, f is continuous on $(0, \infty)$.

Appeared on: F19

Problem 66. **(*)** Consider the function $f(x) = \sum_{k=1}^{\infty} (1 - \cos(x/k))$.
You may use without proof the following inequalities in this problem:

$$|\sin t| \leq |t|, \quad |1 - \cos t| \leq \frac{t^2}{2}, \quad t \in \mathbb{R}.$$

(a) Prove that the series for f converges uniformly on every interval of the form $[-M, M]$ in $\mathbb{R}$.

(b) Prove that f is differentiable on $\mathbb{R}$.

Appeared on: S19

Problem 67. Show that the following series converges uniformly on (r, ∞) for any real number $r > 1$.

$$\sum_{n=1}^{\infty} \frac{n \ln(1+nx)}{x^n}$$

Appeared on: W19

Problem 68. (*)

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 + x^4}{n^4 + x^2}.$$

(a) Prove that the series converges uniformly on $[-R, R]$ for any $R > 0$.

(b) Prove that f is continuous on $\mathbb{R}$.

Appeared on: F18

Problem 69. (*) Consider the function

$$f(x) = \sum_{k=0}^{\infty} e^{-kx} \cos kx.$$

(a) Prove that the series converges uniformly on $[a, \infty)$ for any $a > 0$.

(b) Prove that f is a continuous function on $(0, \infty)$.

Appeared on: F17

Problem 70. (*) Consider the series

$$\sum_{n=1}^{\infty} e^{-nx^2} \sin(nx).$$

(a) Prove that this series converges uniformly on $[a, \infty)$, for each $a > 0$.

(b) Does the series converge uniformly on $[0, \infty)$? Justify your answer.

Appeared on: S16

Problem 71. Let $P = \{2, 3, 5, 7, 11, 13, \ldots\}$ be the set of prime numbers.

1. Find the radius of convergence R of the power series

$$f(x) = \sum_{p \in P} x^p = x^2 + x^3 + x^5 + x^7 + \ldots.$$

2. Show that $0 \le f(x) \le \frac{x^2}{1-x}$ for $0 \le x < R$.

Appeared on: F15

Problem 72. (*) Consider

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sin(2^k x).$$

1. Show that f is continuous on $\mathbb{R}$.
2. Show that f is not differentiable at $x = 0$. (*Hint:* Consider the sequence $\{x_n\} = \{\frac{\pi}{2^n}\}$.)

Appeared on: W25

Problem 73. Suppose that $\{a_k\}$ is a sequence with $|a_k| \leq 1$ for all $k \in \mathbf{N}$.

1. Prove that the series $\sum_{k=1}^{\infty} a_k x^k$ and $\sum_{k-1}^{\infty} k a_k x^{k-1}$ converge uniformly and absolutely on any closed interval contained in $(-1, 1)$.
2. Prove that
$$\frac{d}{dx}\left(\sum_{k=1}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$
for all $x \in (-1, 1)$.

Appeared on: F24

Problem 74. Let $a > 0$ and define $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}(ax)^n$.

1. Find the interval of convergence.
2. Let $0 < c < R$ where R is the radius of convergence. Show the convergence is uniform on $[-c, c]$.

Appeared on: S24

Problem 75. **(*)** Let
$$f(x) = \sum_{n=1}^{\infty} \frac{n^x}{3^n - 7}.$$
Show f is continuous on $[0, \infty)$.

Appeared on: W24

Problem 76. **(*)** Show that $f(x) = \sum_{n=1}^{\infty} \arctan\left(\frac{x}{n^2}\right)$ is a continuous function on all of $\mathbf{R}$.

Appeared on: F23

Problem 77. **(*)** Prove that the series
$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 + x^4}{n^4 + x^2}$$
converges to a continuous function $f : \mathbf{R} \to \mathbf{R}$.

Appeared on: W23

Problem 78. **(*)** Prove that
$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$$
is continuous on $\mathbf{R}$.

Appeared on: F22

Problem 79. (*) Let

$$f_n(x) = \frac{x}{(x + \cos(x/n))^n}$$

for each $n = 1, 2, 3, \ldots$. Prove that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is continuous on $[1, 2]$.

Appeared on: S22

Problem 80. Let (f_n) be a sequence of increasing functions on $[a, b]$ with $\sum_{n=1}^{\infty} f_n(x)$ absolutely convergent when $x = a$ and when $x = b$. Show that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for every $x \in [a, b]$ and that also the series converges uniformly on $[a, b]$.

Integration

Appeared on: S21

Problem 81.(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and assume that the lower integral of f on $[a,b]$ is positive. Show that there exists an interval $[c,d] \subseteq [a,b]$ with $c < d$ with $f(x) > 0$ for $x \in [c,d]$.

Appeared on: W21

Problem 82.(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Prove that if f is continuous on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0).$$

Appeared on: F20

Problem 83. (*)

(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and monotonically increasing, with $f(0) = 0$, $f(1/2) = 1$, and $f(1) = 2$. Prove that

$$\int_0^1 f(x)\, dx > 1/2.$$

Appeared on: W20

Problem 84. Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)x^n\, dx = 0.$$

Appeared on: F19

Problem 85.(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Let a_n be a positive sequence of real numbers converging to 0 and let $B = \{b_1, b_2, b_3, \ldots\}$ be a countably infinite subset of $[0,1]$. Consider the function f on $[0,1]$ defined by

$$f(x) = \begin{cases} a_n, & x = b_n \\ 0, & x \notin B \end{cases}.$$

Use your definition from (a) to prove that f is Riemann integrable on $[0,1]$.

Appeared on: W19

Problem 86.(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Let f be bounded on $[a,b]$ and assume that there exists a partition P with $L(f,P) = U(f,P)$. Use the definition of Riemann integrability to characterize f.

Appeared on: F18

Problem 87.(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function with the property that f is Riemann integrable on $[a,c]$ for all $a < c < b$. Use the definition of Riemann integrability to show that f is Riemann integrable on $[a,b]$.

Appeared on: S18

Problem 88.(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Use your definition from (a) to prove that if $f : [a,b] \to \mathbb{R}$ is continuous and

$$\int_a^b |f(x)| \, dx = 0,$$

then $f(x) = 0$ for all $x \in [a,b]$.

Appeared on: F17

Problem 89. (*)

(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Use your definition from (a) to prove that

$$f(x) = \begin{cases} 1, & x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

is integrable on $[0,1]$ and compute the value of the integral $\int_0^1 f(x)\, dx$.

Appeared on: S17

Problem 90. (*)

(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Use the definition of the Riemann integral to prove that $f(x) = \frac{1}{1+x}$ is Riemann integrable on $[0,b]$, for any $b > 0$.

Appeared on: F16

Problem 91.(a) State the definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a,b]$.

(b) Let

$$g_n(x) = \begin{cases} n & : \quad 0 \le x \le 1/n \\ 0 & : \quad 1/n < x \le 1 \end{cases}$$

and let f be any continuous function on $[0, 1]$. Use the definition of the Riemann integral to compute

$$\lim_{n\to\infty} \int_0^1 f(x)g_n(x)\,dx \text{ in terms of } f.$$

Appeared on: F15

Problem 92.(a) State the definition for a real valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a, b]$.

(b) Let $f : [a, b] \to \mathbb{R}$ be increasing on the inverval $[a, b]$. Use the definition to prove that f is Riemann integrable on $[a, b]$.

Appeared on: W24

Problem 93. (*)

1. State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable.
2. Let $f : [a, b] \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0 & x \in [a, b] \cap \mathbf{Q} \\ x & x \in [a, b] \backslash \mathbf{Q} \end{cases}.$$

Use your definition to decide with proof if f is Riemann integrable.

Appeared on: F23

Problem 94. (*)

1. State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on $[a, b]$.
2. Use this definition to prove that the function f defined on $[0, \pi/2]$ by

$$f(x) = \begin{cases} \cos^2 x & x \in \mathbf{Q} \\ 0 & \text{otherwise} \end{cases}$$

is not Riemann integrable.

Appeared on: W23

Problem 95. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous, with

$$\int_0^1 f(xt)\,dt = 0 \text{ for all } x \in \mathbf{R}.$$

Show that $f(x) \equiv 0$.

Appeared on: F22

Problem 96. 1. State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on $[a, b]$.

2. Let $f : [a,b] \to \mathbf{R}$ be Riemann integrable. Prove that $|f(x)|$ is also Riemann integrable and that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Appeared on: W22

Problem 97. (*)

1. State a definition for a real valued function $f : [a,b] \to \mathbf{R}$ to be Riemann integrable on $[a,b]$.

2. Let

$$f(x) = \begin{cases} 1, & 1 \leq x < 2 \\ 10, & x = 2 \\ 2, & 2 < x \leq 3. \end{cases}$$

Use your definition to prove that f is integrable on $[1,3]$.

Appeared on: S15

Problem 98. 1. State a definition for a real valued function $f : [a,b] \to \mathbf{R}$ to be Riemann integrable on $[a,b]$.

2. Let $f : [a,b] \to \mathbf{R}$ be a continuous function. Use your definition to prove that f is integrable on $[a,b]$.

Appeared on: S14

Problem 99. 1. State a definition for a real valued function $f : [a,b] \to \mathbf{R}$ to be Riemann integrable on $[a,b]$.

2. Let f, g be Riemann integrable functions and suppose that the set E is finite where

$$E = \{x \in (a,b) : f(x) \neq g(x)\}.$$

Use your definition of Riemann integrability to show that $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$.
Hint: Consider the function $f - g$.

Topology of **R***, and sets*

Problem 100. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, periodic function. Prove that the set $f(\mathbb{R})$ is compact. (Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* if there exists a nonzero constant P such that $f(x) = f(x + P)$ for all $x \in \mathbb{R}$.)

Appeared on: W20