FORMAL CONSTRUCTIONS IN THE BRAUER GROUP OF THE FUNCTION FIELD OF A p-ADIC CURVE

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Abstract. We study the relationship between the cohomology of the function field of a curve over a complete discretely valued field and that of the function ring of curves resulting over its residue field. The results are applied to prove the existence of noncrossed product division algebras and indecomposable division algebras of unequal period and index over the function field of any p-adic curve, generalizing the results and methods of [10].

1. Introduction

Let F be a field. An *F-division algebra* D is a division ring that is finite-dimensional over F and whose center is F. We say D is a *crossed product* if D contains a Galois field extension E/F that is maximal in D with respect to containment of subfields, and a *noncrossed product* if it does not. D is a crossed product precisely when its multiplication rule can be described by a Galois 2-cocycle with values in the multiplicative group E^* of a Galois extension E/F, and as a result all early division algebra constructions were crossed products by default. The classical fact that every F-division algebra contains a *separable* maximal subfield implies that the matrix algebra $\mathrm{M}_n(D)$ is a crossed product for some n, for any D. During the development of class field theory it was established that all $\mathbb{Q}$ and $\mathbb{Q}_p$-division algebras are crossed products via a *cyclic* Galois extension. Decades later, noncrossed product F-division algebras were shown to exist, in a 1972 paper by Amitsur ([1]). Since then several different and interesting constructions have appeared over various fields (see [5, Section 0] for additional background).

An F-division algebra is *decomposable* if it can be expressed as an F-tensor product of two (nontrivial) F-division algebras, and *indecomposable* otherwise. Division algebras admit two measures of size called *period* and *index*, analogous to exponent and order for finite abelian groups. Before it was proved otherwise, it would have been reasonable to conjecture that any F-division algebra of unequal period and index would decompose into factors of equal period and index. Indecomposable division algebras of unequal period and index first appeared in 1979, in papers by Amitsur-Rowen-Tignol [3] and Saltman [37]. Since then there have been several other constructions, notably by Jacob in [25] and Karpenko in [26, 27] (see [5, Section 9]).

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F-division algebras are fundamentally arithmetic objects in the sense that they become trivial when scalar-extended to an algebraic closure of F. As such their taxonomy is a reflection of F's arithmetic, and this motivates our interest in constructions of noncrossed products and indecomposables of unequal period and index. We are particularly interested in the relationship between possible constructions over arithmetically related fields, such as the function fields of curves over discretely valued fields and over their residue fields.

The *Brauer group* $\mathrm{Br}(F)$ is a group formed by isomorphism classes of F-division algebras, and its n-torsion subgroup is isomorphic (via the crossed product construction) to the degree-two cohomology group $\mathrm{H}^2(F,\mu_n)$, for n prime-to-char(F). In this paper we study the cohomology groups $\mathrm{H}^q(F,(\mathbb{Z}/n)(i))$ of a field F that is finitely generated and of transcendence degree one over a complete discretely valued field $K=(K,v)$, and in particular over the p-adic field $\mathbb{Q}_p$. Such a field F is always the function field of a regular, projective, flat relative curve X/O_v. In [10] it was shown that if $K=\mathbb{Q}_p$ and F admits a smooth model $X/\mathbb{Z}_p$ then there exist noncrossed product F-division algebras, and indecomposable F-division algebras of unequal prime-power period and index. This applies to fields such as $\mathbb{Q}_p(t)$ but not to the function fields of all p-adic curves. In this paper we generalize the machinery of [10] to the function field F of an arbitrary p-adic curve, and then prove the existence of noncrossed product F-division algebras, and indecomposable F-division algebras of unequal prime-power period and index.

The Brauer groups of function fields of p-adic curves have been the focus of several important papers recently, including the work by Saltman [39, 41], which initiated much of the recent activity, work by Harbater-Hartmann-Krashen [23, 24], which uses patching methods, and work by Lieblich [30] and Parimala-Suresh [34] (on the u-invariant of quadratic forms). The use of 2-dimensional models to study these Brauer groups goes back at least to Saito [36] and Kato [28], and ultimately to Grothendieck.

Our technique is to lift constructions from the rational function ring $\kappa(C)$ of the reduced scheme C underlying the closed fiber of $X/\mathbb{Z}_p$, which is a product of global fields. Our main technical advance over [10] is to show how to do this if C is reducible with mild singularities. This situation can be unavoidable, and arises naturally after blowups of X. The extra generality has crucial implications for the theory of $\mathrm{Br}(F)$, and allows us, for example, to prove in [11] that every F-division algebra of prime period $\ell\neq p$ and index ℓ^2 decomposes into two cyclic F-tensor factors, hence is a crossed product, even when $F=\mathbb{Q}_p(t)$. The latter result generalizes Suresh's result [42], which assumes roots of unity, and does not cover $\mathbb{Q}_p(t)$ in general.

We summarize the technical results. Let $K=(K,v)$ be a complete discretely valued field K with residue field k, n a number prime-to-char(k), and let F/K be a finitely generated field extension of transcendence degree one. Let X/O_v be a regular, projective, flat, 2-dimensional model for F. The closed fiber X_0/k is a projective curve, whose underlying reduced structure C may be assumed to consist of regular irreducible components C_i, exactly two of which meet at the singular points $\mathcal{S}$ of C. Our main theorem (Theorem 4.9) constructs for any integer r and any $q \geq 0$ a homomorphism

$$\lambda : \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r)) \longrightarrow \mathrm{H}^q(\mathrm{O}_{X,\mathcal{S}}, \mathbb{Z}/n(r)) \hookrightarrow \mathrm{H}^q(F, \mathbb{Z}/n(r))$$

whose first arrow splits the restriction map $\mathrm{H}^q(\mathrm{O}_{X,\mathcal{S}}, \mathbb{Z}/n(r)) \to \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r))$. Let $F_C = \prod_i F_{C_i}$ be the product of the completions of F with respect to the valuations defined by the C_i. We use λ to construct for any $q \geq 1$ a commutative diagram

$$\begin{array}{ccc} & \overset{\lambda}{\nearrow} & \mathrm{H}^q(F, \mathbb{Z}/n(r)) \\ & & \downarrow \mathrm{res}_{F|F_C} \\ \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r)) \oplus \mathrm{H}^{q-1}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r-1)) & \xrightarrow[\eta]{} & \mathrm{H}^q(F_C, \mathbb{Z}/n(r)) \end{array}$$

so that $\lambda((\alpha_C, \theta_C)) = \lambda(\alpha_C) + (\pi) \cdot \lambda(\theta_C)$ for a uniformizer π of F_C. When $K = \mathbb{Q}_p$, $q = 2$, and $r = 1$, we show that λ factors through $\mathrm{H}^2(F_C, \mu_n)$ to determine an index-preserving map from the subgroup $\mathrm{im}(\eta) \leq \mathrm{H}^2(F_C, \mu_n)$ to $\mathrm{H}^2(F, \mu_n)$, splitting the restriction map. Since the residue fields $\kappa(C_i)$ in this case are global fields we are then able to construct indecomposable F-division algebras and noncrossed product F-division algebras, in the same manner as [10]. When the dual graph of C has nontrivial topology, i.e., nonzero (first) Betti number, we construct cyclic covers of X that are (defined and) trivial at every point of X except the generic point of X. These arise as cyclic covers of C that are trivial at every point, and transported to X via λ. When $K = \mathbb{Q}_p$ they are the completely split cyclic covers of Saito ([36]). We thank Colliot-Thélène for drawing our attention to these interesting examples.

2. Background and Conventions

We use [31, Chapter 8,9], [22, Section 2], and [21, Chapter XIII] for many of the following definitions.

2.1. General Conventions. Let S be an excellent scheme, n a number that is invertible on S, and for any $i \in \mathbb{Z}$ let $\Lambda = (\mathbb{Z}/n)(i)$ be the twisted étale sheaf. Let $K(S)$ denote the total fraction ring of S, which is the ring of global sections of the sheaf of total fractions. We write $\mathrm{H}^q(S, \Lambda)$ for the étale cohomology group, and if Λ is understood (or doesn't matter) we write $\mathrm{H}^q(S, r)$ instead of $\mathrm{H}^q(S, \Lambda(r))$, and $\mathrm{H}^q(S)$ in place of $\mathrm{H}^q(S, 0)$. If $S = \operatorname{Spec} A$ for a ring A then we write $\mathrm{H}^q(A, r)$. If D is a closed subscheme of S we write $\kappa(D)$ for its total fraction ring. If $T \to S$ is a morphism of schemes then the restriction $\mathrm{res}_{S|T} : \mathrm{H}^q(S) \to \mathrm{H}^q(T)$ is defined, and we write $\beta_T = \mathrm{res}_{S|T}(\beta)$. If $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$ for a ring B we write $\beta_B = \mathrm{res}_{A|B}(\beta)$ instead. If $Z \to S$ is another morphism, we write Z_T for the fiber product $Z \times_S T$.

2.2. Valuation Theory. If v is a discrete valuation on a field F we write $\kappa(v)$ for the residue field of the valuation ring O_v, and F_v for the completion of F at v. If S is a noetherian connected normal scheme with function field F and v arises from a prime divisor D on S, we generally substitute D for v, and write v_D for v, $\kappa(D)$ for $\kappa(v)$, and F_D for F_v. If D is a sum of prime divisors D_i we write $F_D = \prod_i F_{D_i}$. We will abuse this notation and write F_D should D be a closed subscheme of pure codimension one.

By "Witt's Theorem" (see [16, 7.10]) for each discrete valuation v on the field F there is a short exact sequence

$$0 \longrightarrow \mathrm{H}^q(\kappa(v),\Lambda) \xrightarrow{\text{inf}} \mathrm{H}^q(F_v,\Lambda) \xrightarrow{\partial_v} \mathrm{H}^{q-1}(\kappa(v),\Lambda(-1)) \longrightarrow 0$$

where inf is the *inflation* map and ∂_v is the *residue* map. We use this sequence to identify $\mathrm{H}^q(\kappa(v),\Lambda)$ with the corresponding subgroup of $\mathrm{H}^q(F_v,\Lambda)$. If $\alpha \in \mathrm{H}^q(F_v,\Lambda)$ then $\partial_v(\alpha)$ is called the *residue* of α at v. More generally, suppose T is a noetherian scheme, ξ is a generic point of T, and $\alpha \in \mathrm{H}^q(T,\Lambda)$. Then for each discrete valuation v on the field $F = \kappa(\xi)$, α has a residue

$$\partial_v(\alpha) \overset{\text{df}}{=} \partial_v(\alpha_{F_v}) \in \mathrm{H}^{q-1}(\kappa(v),\Lambda(-1))$$

We say α is *unramified at* v if $\partial_v(\alpha) = 0$ and *ramified at* v if $\partial_v(\alpha) \neq 0$. If α is unramified at v the *value* of α at v is the element

$$\alpha(v) = \alpha_{F_v} \in \mathrm{H}^q(\kappa(v),\Lambda) \leq \mathrm{H}^q(F_v,\Lambda)$$

(see [16, 7.13, p.19]). Suppose T is as above and $T \to S$ is a birational morphism of noetherian schemes (see [17, I.2.2.9]). The *ramification locus of* $\alpha \in \mathrm{H}^q(T,\Lambda)$ *on* S_{red} is the sum (disregarding multiplicities) of the prime divisors on S_{red} that determine valuations at which α is ramified, over all generic points of S_{red}.

2.3. Tamely Ramified Covers. Let S be a noetherian normal scheme, D a divisor on S, $U = S - D$, and for each generic point ξ of $\operatorname{Supp} D$, let $K_\xi = \operatorname{Frac} \mathrm{O}_{S,\xi}$. Note that S being normal implies that the irreducible components of S are disjoint. We say $\rho : T \to (S, D)$ is a *tamely ramified cover (ramified along D)* if

1. $\rho_U : V = T \times_S U \to U$ is étale;
2. $\rho : T \to S$ is finite;
3. each irreducible component of T dominates an irreducible component of S;
4. T is normal;
5. for each generic point ξ of $\operatorname{Supp} D$, the étale K_ξ-algebra L defined by $\operatorname{Spec} L = V \times_U \operatorname{Spec} K_\xi$ is tamely ramified with respect to the discrete valuation determined by ξ.

In (5) L/K_ξ is étale, hence L is a finite product of separable field extensions of K_ξ, and L is tamely ramified if each field extension is tamely ramified (with respect to $\mathrm{O}_{S,\xi}$) in the usual sense. If $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$, we will also say B is a *finite tamely ramified A-algebra.* This is the definition [22, Definition 2.2.2] given by Grothendieck-Murre. If S is regular and D has normal crossings on S, the last condition (5) is equivalent to the definition [21, XIII.2.3(c)] given in SGA 1, allowing us to use the results of [21, XIII.5].

If S is a noetherian scheme whose irreducible components are normal, we will say $\rho : T \to (S, D)$ is a tamely ramified cover (ramified along D) if again ρ is finite and $V \to U = S - D$ is étale, and the restriction ρ_{S_i} to each irreducible component S_i of S is a tamely ramified cover (ramified along D_{S_i}).

2.4. Standard Setup. $K = (K, v)$ is a complete discretely valued field with complete discrete valuation ring $R = \mathrm{O}_v$ and residue field k, F/K is a finitely generated field extension of transcendence degree, and X is a regular connected 2-dimensional scheme that is flat and projective over $\operatorname{Spec} R$ and has function field $F = K(X)$. We call X/R a (regular) *relative curve*, and write $X_0 = X \otimes_R k$ for the closed fiber (a projective curve over k by [31, 8.3.3]), $C = X_{0,\mathrm{red}}$ for the reduced subscheme underlying X_0, and $C_1, \dots, C_m$ for the irreducible components of C. We assume each C_i is regular, and at most two of them meet at any closed point of X, a situation that can always be obtained by blowing up, using Lipman's embedded resolution theorem (see [31, 9.2.4]). We let $\mathcal{S}$ denote the set of singular points of C. Note that all closed points of X lie on C, and for each closed point $z \in X$ we have $\dim \mathrm{O}_{X,z} = 2$ by [31, 8.3.4(c)]. Since X is regular, $\mathrm{O}_{X,z}$ is then factorial by Auslander-Buchsbaum's theorem.

We say an effective (Cartier) divisor D on our relative curve X/R is *horizontal* if each irreducible component of $\operatorname{Supp} D$ maps surjectively (hence finitely by [31, 8.3.4(a),(b)]) to $\operatorname{Spec} R$, and *vertical* if $\operatorname{Supp} D$ is contained in C. If D is a reduced and irreducible horizontal divisor then it is flat over $\operatorname{Spec} R$, since R is a discrete valuation ring. Every effective divisor on a regular relative curve X/R is a sum of horizontal and vertical divisors, and the horizontal prime divisors are exactly the closures of the closed points of the generic fiber ([31, 8.3.4(b)]). Since R is henselian, each irreducible horizontal divisor has a single closed point. More generally, since R is henselian and X/R is projective there is a 1-1 correspondence between connected components of a horizontal divisor D and connected components of $D \otimes_R k$ ([19, Proposition 18.5.19]).

2.5. Distinguished Divisors. In general there will be many horizontal divisors on our X/R that restrict to a given divisor on C. In order to construct our lifts from C to X we select a single regular horizontal divisor for each closed point, as follows.

Proposition 2.6. *Assume the setup of* (2.4). *Then for each closed point $z \in X$ there exists a regular irreducible effective horizontal divisor $D \subset X$ that intersects each irreducible component of C passing through z transversally at z.*

Proof. Transversal intersection with a single component is by [31, 8.3.35(b)] and its proof (see also [20, 21.9.12]). Thus if $z \in C_i \cap C_j$ $(i \neq j)$ and t_i and t_j are local equations for C_i and C_j, then we have local equations f_i and f_j for effective regular horizontal divisors such that $(f_i, t_i) = (f_j, t_j) = \mathfrak{m}_z \subset \mathrm{O}_{X,z}$. If $(f_j, t_i) = \mathfrak{m}_z$ or $(f_i, t_j) = \mathfrak{m}_z$ then a suitable D is defined locally by f_j or f_i. Otherwise $(f_i + f_j, t_i) = (f_i + f_j, t_j) = \mathfrak{m}_z$, and we define D locally by $f_i + f_j$. The rest of the proof proceeds as in [31, 8.3.35]. □

We fix a set of these (prime) divisors, and let $\mathscr{D}$ denote the union of their supports. We will say a divisor D is *distinguished* and write $D \in \mathscr{D}$ whenever D is reduced and supported in $\mathscr{D}$. Let $\mathscr{D}_{\mathcal{S}}$ denote the subset that *avoids* the set of singular points $\mathcal{S}$ of C. Note that each $D \in \mathscr{D}$ is a disjoint union of its irreducible components, each of which meets each irreducible component of C transversally.

3. Structure of Tame Covers

Lemma 3.1 (Structure). *Assume the setup of (2.4). Suppose $\rho : Y \to (X, D)$ is a tamely ramified cover, where $D \in \mathscr{D}$. Then*

a) *The structure map $\rho : Y \to X$ is flat.*

b) *Y/R is a regular relative curve, $Y_{0,\mathrm{red}} = C_Y$, each irreducible component of C_Y is regular, $\mathcal{S}_Y = \rho^{-1}(\mathcal{S})$ is the set of singular closed points of C_Y, and exactly two irreducible components of C_Y meet at each point of $\mathcal{S}_Y$.*
c) *The support of the irreducible components of D'_Y for $D' \in \mathscr{D}$ generate a set $\mathscr{D}_Y$ of distinguished divisors on Y.*

Proof. Since $Y \to X$ is finite, $\dim(X) = \dim(Y) = 2$ by [31, 2.5.10], and $Y \to \operatorname{Spec} R$ is projective as the composition of projective morphisms ([31, 3.3.32]). Let $y \in Y$ be a closed point and set $x = f(y)$, $A = \mathrm{O}_{X,x}$, $B' = \mathrm{O}_{Y,x}$, and $B = \mathrm{O}_{Y,y}$. By B' we mean the ring of the fiber over $\operatorname{Spec} \mathrm{O}_{X,x}$ on Y. Choose a geometric point over x that lifts to each point of Y lying over x, and use this in the following to define the strict henselizations with respect to the maximal ideals of these points.

Since the statements involving D are local and D is a disjoint union of its irreducible components we may assume D is irreducible. Let $C_i \subset C$ be a (regular) irreducible component going through x, and let $\{f, t\} \subset A$ be the regular system of parameters formed by local equations for the distinguished prime divisor passing through x, and for C_i, respectively. Then the strict henselization A^{sh} of A with respect to the maximal ideal of A is a two-dimensional regular local ring, faithfully flat over A, with regular system of parameters $\{f, t\}$ (see [20, 18.8]).

If $x \notin D$ then $B' \otimes_A A^{\mathrm{sh}}$ is a finite étale A^{sh}-algebra by base change, since $\rho|_{X-D}$ is finite-étale. If $x \in D$ then $B' \otimes_A A^{\mathrm{sh}}$ is a finite tamely ramified A^{sh}-algebra by [22, Lemma 2.2.8]. By [20, 18.8.10], B^{sh} is a factor of the direct product decomposition of $B' \otimes_A A^{\mathrm{sh}}$, hence B^{sh} is a finite tamely ramified local A^{sh}-algebra, in particular it is a normal local ring, hence it is a normal domain. It follows that B^{sh} is the integral closure of A^{sh} in the field $\widetilde{L} \overset{\mathrm{df}}{=} \operatorname{Frac} B^{\mathrm{sh}}$. Since the tame fundamental group of the strictly henselian regular local ring A^{sh} is abelian ([21, XIII.5.3]) the field extension $\widetilde{L}/\operatorname{Frac} A^{\mathrm{sh}}$ is Galois, and by Abhyankar's Lemma ([13, A.I.11], see also [22, Corollary 2.3.4])

$$B^{\mathrm{sh}} = A^{\mathrm{sh}}[T]/(T^e - f) \quad (\text{some } e \geq 1)$$

By [22, Lemma 1.8.6] B^{sh} is a regular (2-dimensional) local ring with system of parameters $\{\sqrt[e]{f}, t\}$. Since $B \to B^{\mathrm{sh}}$ is faithfully flat and B^{sh} is regular, B is regular by flat descent ([18, 6.5.1] or [32, 23.7(i)]), and since B is the local ring of an arbitrary closed point, we conclude Y is regular. It follows that $\rho : Y \to X$ is flat by [32, 23.1], proving (a), and since Y is regular and $Y \to \operatorname{Spec} R$ is flat and projective, Y/R is a regular relative curve.

We derive a system of parameters for B. The prime ideal $(\sqrt[e]{f}) \subset B^{\mathrm{sh}}$ is the only one lying over $(f)A^{\mathrm{sh}}$ since, for $\kappa(f) = \operatorname{Frac} A^{\mathrm{sh}}/(f)A^{\mathrm{sh}}$, the ring $B^{\mathrm{sh}} \otimes_{A^{\mathrm{sh}}} \kappa(f) = \kappa(f)[T]/(T^e)$ of the fiber over $\operatorname{Spec} \kappa(f)$ consists of a single prime ideal. Since $\operatorname{Spec} B^{\mathrm{sh}} \to \operatorname{Spec} B$ is surjective, the image $(\sqrt[e]{f}) \cap B$ of $(\sqrt[e]{f})$ in $\operatorname{Spec} B$ is the unique prime lying over $(f) \subset A$, and since $B \to B^{\mathrm{sh}}$ is flat, $(\sqrt[e]{f}) \cap B$ has height one, hence is principal (since B is factorial), hence $(\sqrt[e]{f}) \cap B = (g)$ for some $g \in B$. Then $(\sqrt[e]{f})$ is the unique prime of B^{sh} lying over (g), and (g) is the unique prime of B lying over (f). Since $B \to B^{\mathrm{sh}}$ is unramified, $(g)B^{\mathrm{sh}} = (\sqrt[e]{f})$. Since $B \to B^{\mathrm{sh}}$ is faithfully flat, $IB^{\mathrm{sh}} \cap B = I$ for all ideals I of B (by e.g. [4, Exercise 3.16]), so since $(g, t)B^{\mathrm{sh}} = (\sqrt[e]{f}, t)$ is maximal, $(g, t)B^{\mathrm{sh}} \cap B = (g, t)$ is the maximal ideal of B. Thus $\{g, t\}$ is a regular system for B.

Since t is a local equation for $\rho^{-1}C_i$, $\rho^{-1}C_i$ is regular and irreducible at y for each C_i passing through x. In particular $C_Y = \bigcup_i \rho^{-1}C_i$ is reduced, and so equals $Y_{0,\mathrm{red}}$, and since at most two irreducible components of C meet at x, at most two irreducible components of C_Y meet at y. Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, C_Y has at least the number of components at y as does C at x. Thus y is a singular point on C_Y if and only if $x = f(y) \in \mathcal{S}$. This completes the proof of (b).

If $D' \in \mathscr{D}$ is the distinguished (horizontal) prime divisor running through x then there is a single irreducible component of D'_Y passing through y, whose support $D'_{Y,\mathrm{red}}$ has local equation g at y. Thus each irreducible component of D'_Y covers D', hence $\operatorname{Spec} R$, hence D'_Y is horizontal. Since g is part of the regular system $\{g,t\}$ at the arbitrary closed point y we see that $D'_{Y,\mathrm{red}}$ is regular, and since t is a local equation for an arbitrary irreducible component of C_Y passing through y, $D'_{Y,\mathrm{red}}$ intersects each component of C_Y transversally. Thus the supports of the irreducible components of the various D'_Y for $D' \in \mathscr{D}$ generate a set of distinguished divisors $\mathscr{D}_Y$ for Y. This proves (c). □

Lemma 3.2. *Suppose X is a regular noetherian scheme and L is an étale $K(X)$-algebra that is tamely ramified along a divisor D. Then the normalization Y of X in L defines a tamely ramified cover $\rho : Y \to (X,D)$.*

Proof. Since X is regular, its connected components are integral regular schemes, hence we may assume X is integral. Since $L/K(X)$ is étale, L is a product of finite separable field extensions of $K(X)$, hence we may assume $L/K(X)$ is itself a finite separable field extension. Then the normalization Y exists, Y is normal by definition, and $\rho : Y \to X$ is finite by [31, 4.1.25]. Since $\rho : Y \to X$ induces an injection $K(X) \to K(Y)$, Y dominates X. Let $U = X - D$, and set $V = Y \times_X U$. Since X is normal, Y is connected, and $\rho|_V$ is unramified, $\rho|_V$ is étale by [21, I.9.11] (see also [33, I.3.20]). Therefore $Y \to (X,D)$ is a tamely ramified cover. □

The next lemma shows how distinguished divisors split in tamely ramified covers.

Lemma 3.3. *Assume the setup of (2.4). Suppose $\rho : Y \to (X,D)$ is a tamely ramified cover, where $D \in \mathscr{D}$, and $D' \in \mathscr{D}_{\mathcal{S}}$ is irreducible. Suppose $E \subset D'_{Y,\mathrm{red}}$ is a distinguished prime divisor lying over D' as in Lemma 3.1(c), $y = E \times_Y C_Y$, and $x = D' \times_X C$. Then y and x are regular closed points, and the ramification (resp. inertia) degree of v_E over $v_{D'}$ equals the ramification (resp. inertia) degree of v_y over v_x.*

Proof. Since we assume (2.4) and $D \in \mathscr{D}$ we have Lemma 3.1, which shows C_Y is reduced and $E \subset D'_{Y,\mathrm{red}}$ is distinguished. Note that either $D' \cap D = \varnothing$ or $D' \subset D$. Since D' and E are distinguished and avoid the singular points of C and C_Y, they intersect the reduced closed fibers C and C_Y transversally, hence $x = D' \times_X C$ and $y = E \times_Y C_Y$ are regular closed points. We must show that $[\kappa(E) : \kappa(D')] = [\kappa(y) : \kappa(x)]$ and that $v_E(f) = v_y(f_0)$, where $f \in \mathrm{O}_{X,D'}$ is a local equation for D' on X and $f_0 \in \mathrm{O}_{C,x}$ is a local equation for x on C.

Since D' is horizontal and irreducible, $D' = \operatorname{Spec} S$ for S a finite local R-algebra by [31, 8.3.4] and [33, I.4.2], and S is a discrete valuation ring since D' is regular. The map $E \to \rho^{-1}D' \to D'$ is finite as a composition of finite morphisms, hence $E = \operatorname{Spec} T$ for T a finite local S-algebra, again a discrete valuation ring since E

is regular. Since S is a discrete valuation ring, $S \to T$ is finite, and T is torsion-free, T is a free S-module of finite rank, and so $[T : S]$ is well defined. Since the generic point of E lies over that of D', we have $\operatorname{Frac} T = T \otimes_S \operatorname{Frac} S$, hence $[\kappa(E) : \kappa(D')] = [T : S]$.

Let $A = \mathrm{O}_{X,x}$, $B = \mathrm{O}_{Y,y}$, let t be a local equation for C at x, and set $A_0 = A/(t)$ and $B_0 = B/(t)B$, the (reduced) local rings of the fibers through x and y, as in the proof of Lemma 3.1. Already $\kappa(x) = S \otimes_A A_0$ and $\kappa(y) = T \otimes_B B_0$ by the transversality of the intersections. Since $B_0 = B \otimes_A A_0$ we have $\kappa(y) = T \otimes_A A_0$, hence $[\kappa(y) : \kappa(x)] = [T : S] = [\kappa(E) : \kappa(D')]$ by base change.

Let $f \in A$ and $g \in B$ be defined as in the proof of Lemma 3.1. To compute the ramification degree, note that since $B \to B^{\mathrm{sh}}$ is faithfully flat, $(g^e)B = (g^e)B^{\mathrm{sh}} \cap B = (f)B^{\mathrm{sh}} \cap B = (f)B$, hence $g^e = fu$ for some $u \in B^*$. Since f and g are uniformizers for $v_{D'}$ and v_E, respectively, it follows that $e(v_E/v_{D'}) = v_E(f) = e$. On the other hand, let f_0 be the image of f in A_0, and let g_0 be the image of g in B_0. Then f_0 cuts out the closed point x on C and g_0 cuts out y on C_Y by transversality. Thus f_0 and g_0 are uniformizers for v_x and v_y, and since $g_0^e = f_0 u_0$, where u_0 is the image of u in B_0^*, we have $e(v_y/v_x) = v_y(f_0) = e$, as desired. This completes the proof. □

4. Lifting Cohomology Classes

4.1. Let k be a field, and let C/k be a reduced connected projective curve with regular irreducible components $C_1, \ldots, C_m$, at most two of which meet at any closed point. Denote the singular points of C by $\mathcal{S}$ and write $\mathrm{O}_{C,\mathcal{S}}$ for the semilocal ring $\varinjlim_U \mathrm{O}_C(U)$, where U varies over (dense) open subsets of C containing $\mathcal{S}$. Then since C has no embedded points, $\mathrm{O}_{C,\mathcal{S}}$ is a subring of the rational function ring $\kappa(C) = \prod_i \kappa(C_i)$ by [31, 7.1.9]. For each $z \in \mathcal{S} \cap C_i$, let $K_{i,z} = \operatorname{Frac} \mathrm{O}^{\mathrm{h}}_{C_i,z}$, a field since z is a normal point of C_i, and if $\alpha_i \in \mathrm{H}^q(\kappa(C_i))$, let $\alpha_{i,z}$ denote the image in $\mathrm{H}^q(K_{i,z})$.

Since C/k is projective it is separated (over $\mathbb{Z}$), hence if $U \subset C$ is an affine open subset of C then $U \to C$ is an affine map by [31, 3.3.6]. The inverse limit $\varprojlim U$ over affine open subschemes of C containing $\mathcal{S}$ is then a scheme by [19, 8.2.3], and this scheme is $\operatorname{Spec} \mathrm{O}_{C,\mathcal{S}}$ by [31, Exercise 5.1.17(c)]. Therefore $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \Lambda) = \varinjlim \mathrm{H}^q(U, \Lambda)$ by [33, III.1.16].

Lemma 4.2 (Gluing)**.** *Assume the setup of (4.1). There exists an element* $\alpha \in \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \Lambda)$ *that restricts to* $\alpha_C = (\alpha_1, \ldots, \alpha_m) \in \bigoplus_i \mathrm{H}^q(\kappa(C_i), \Lambda)$ *if and only if* α_i *is unramified at each* $z \in \mathcal{S} \cap C_i$*, and* $\alpha_{i,z} = \alpha_{j,z}$ *(as elements of* $\mathrm{H}^q(\kappa(z), \Lambda)$*) whenever* $z \in C_i \cap C_j$.

Proof. There is an exact sequence ([33, III.1.25])

$$
\begin{aligned}
0 \longrightarrow \mathrm{H}^0_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^0(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^0(\kappa(C)) \longrightarrow \mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \\
\longrightarrow \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^1(\kappa(C)) \longrightarrow \mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \\
\longrightarrow \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^2(\kappa(C)) \longrightarrow \mathrm{H}^3_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow
\end{aligned}
\tag{4.3}
$$

where the maps $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}) \to \mathrm{H}^q(\kappa(C))$ are restrictions. Since $\mathcal{S}$ is a disjoint union of closed points, we have $\mathrm{H}^q_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) = \bigoplus_{z \in \mathcal{S}} \mathrm{H}^q_z(\mathrm{O}_{C,\mathcal{S}}) = \bigoplus_{z \in \mathcal{S}} \mathrm{H}^q_z(\mathrm{O}^{\mathrm{h}}_{C,z})$ by excision

([33, III.1.28, p.93]). Since Λ is a smooth group scheme, $\mathrm{H}^q(\mathrm{O}^{\mathrm{h}}_{C,z}) = \mathrm{H}^q(\kappa(z))$, by the cohomological Hensel's lemma [33, III.3.11(a), p.116]. Since the C_k are regular and exactly two of them meet at any $z \in \mathcal{S}$, we have $\operatorname{Spec} \mathrm{O}^{\mathrm{h}}_{C,z} - \{z\} = \operatorname{Spec}(K_{i,z} \times K_{j,z})$ for some i and j, and an "excised" exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0_z(\mathrm{O}^{\mathrm{h}}_{C,z}) \longrightarrow \mathrm{H}^0(\kappa(z)) \longrightarrow \mathrm{H}^0(K_{i,z} \times K_{j,z}) \longrightarrow \mathrm{H}^1_z(\mathrm{O}^{\mathrm{h}}_{C,z}) \longrightarrow \\ \longrightarrow \mathrm{H}^1(\kappa(z)) \longrightarrow \mathrm{H}^1(K_{i,z} \times K_{j,z}) \longrightarrow \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}) \longrightarrow \\ \longrightarrow \mathrm{H}^2(\kappa(z)) \longrightarrow \mathrm{H}^2(K_{i,z} \times K_{j,z}) \longrightarrow \mathrm{H}^3_z(\mathrm{O}^{\mathrm{h}}_{C,z}) \longrightarrow \end{aligned} \tag{4.4}$$

where the map $\mathrm{H}^q(\kappa(z)) \to \mathrm{H}^q(K_{i,z} \times K_{j,z}) = \mathrm{H}^q(K_{i,z}) \oplus \mathrm{H}^q(K_{j,z})$ is the diagonal map given by inflation from $\kappa(z)$ to the henselian fields $K_{i,z}$ and $K_{j,z}$. Since n is prime-to-p, the map $\mathrm{H}^0(\kappa(z)) \to \mathrm{H}^0(K_{i,z})$ is an isomorphism, so $\mathrm{H}^0_z(\mathrm{O}^{\mathrm{h}}_{C,z}) = 0$, and for $q \geq 1$ we have short exact Witt-type sequences

$$0 \to \mathrm{H}^q(\kappa(z)) \to \mathrm{H}^q(K_{i,z}) \xrightarrow{\partial_z} \mathrm{H}^{q-1}(\kappa(z), -1) \to 0$$

Thus the long exact sequence breaks up into short exact sequences

$$0 \to \mathrm{H}^q(\kappa(z)) \to \mathrm{H}^q(K_{i,z} \times K_{j,z}) \to \mathrm{H}^{q+1}_z(\mathrm{O}^{\mathrm{h}}_{C,z}) \to 0 \qquad (q \geq 0) \tag{4.5}$$

By the compatibility of (4.3) with (4.4) the map $\mathrm{H}^q(\kappa(C_i)) \to \mathrm{H}^{q+1}_z(\mathrm{O}^{\mathrm{h}}_{C,z}) \leq \mathrm{H}^{q+1}_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}})$ factors through $\operatorname{res}_{\kappa(C_i)|K_{i,z}}$. Therefore an element $\alpha_C = (\alpha_1, \dots, \alpha_m) \in \mathrm{H}^q(\kappa(C))$ maps to zero in $\mathrm{H}^{q+1}_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}})$ if and only if each couple $(\alpha_{i,z}, \alpha_{j,z})$ is in the image of some $\bar{\alpha} \in \mathrm{H}^q(\kappa(z))$; i.e., $\alpha_{i,z} = \alpha_{j,z}$, and both are unramified. Thus by the exactness of (4.3), α_C comes from $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}})$ if and only if each α_i is unramified at each $z \in \mathcal{S} \cap C_i$, and $\alpha_{i,z} = \alpha_{j,z}$ whenever $z \in C_i \cap C_j$. □

Suppose C is as in (4.1). Since exactly two irreducible components meet at any $z \in \mathcal{S}$ the *dual graph* G_C is defined, and consists of a vertex for each irreducible component of C and an edge for each singular point, such that an edge and a vertex are incident when the corresponding singular point lies on the corresponding irreducible component ([36, 2.23], see also [31, 10.1.48]). The (first) Betti number for G_C is $\beta_C \overset{\mathrm{df}}{=} \mathrm{rk}(\mathrm{H}_1(G_C, \mathbb{Z})) = N + E - V$, where V, E and N are the numbers of vertices, edges, and connected components of G_C, respectively. Note $N = 1$ in the setup of (4.1).

Lemma 4.6. *Assume the setup of (4.1). Then:*

a) *For any integer r, $\mathrm{H}^1(C, \mathbb{Z}/n(r)) \to \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r))$ is injective.*
b) *The map $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(q-1)) \to \mathrm{H}^q(\kappa(C), \mathbb{Z}/n(q-1))$ is injective for $q = 0, 2$, and for $q = 1$ we have*

$$\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \simeq (\mathbb{Z}/n)^{\beta_C} \oplus \Gamma$$

where $(\mathbb{Z}/n)^{\beta_C}$ is the kernel of $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \to \mathrm{H}^1(\kappa(C), \mathbb{Z}/n)$, and $\Gamma \leq \mathrm{H}^1(\kappa(C), \mathbb{Z}/n)$ is the group of tuples that glue as in Lemma 4.2.

Proof. We suppress the notation for $\Lambda = \mathbb{Z}/n(r)$. Let $\mathcal{F} \subset C - \mathcal{S}$ be a finite set of (regular) closed points, and set $U = C - \mathcal{F}$, a dense open subset containing $\mathcal{S}$. The localization exact sequence is

$$0 \longrightarrow \mathrm{H}^0_{\mathcal{F}}(C) \longrightarrow \mathrm{H}^0(C) \longrightarrow \mathrm{H}^0(U) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{H}^q_{\mathcal{F}}(C) \longrightarrow \mathrm{H}^q(C) \longrightarrow \mathrm{H}^q(U) \longrightarrow \mathrm{H}^{q+1}_{\mathcal{F}}(C) \longrightarrow \cdots$$

By excision we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathcal{F}}(C) \longrightarrow \bigoplus_{z\in\mathcal{F}} \mathrm{H}^0(\mathrm{O}^{\mathrm{h}}_{C,z}) \longrightarrow \bigoplus_{z\in\mathcal{F}} \mathrm{H}^0(K_z) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{H}^q_{\mathcal{F}}(C) \longrightarrow \bigoplus_{z\in\mathcal{F}} \mathrm{H}^q(\mathrm{O}^{\mathrm{h}}_{C,z}) \longrightarrow \bigoplus_{z\in\mathcal{F}} \mathrm{H}^q(K_z) \longrightarrow \mathrm{H}^{q+1}_{\mathcal{F}}(C) \longrightarrow \cdots$$

where $K_z = \operatorname{Frac} \mathrm{O}^{\mathrm{h}}_{C,z}$. Since each z is a regular point, $\mathrm{O}^{\mathrm{h}}_{C,z}$ is a discrete valuation ring, and by [12, Section 3.6] we may replace each $\mathrm{H}^{q+1}_z(C)$ with $\mathrm{H}^{q-1}(\kappa(z), -1)$, and the map from $\mathrm{H}^q(U)$, which factors through each $\mathrm{H}^q(K_z)$, is then the residue map ∂_z. We conclude $\mathrm{H}^0(C) = \mathrm{H}^0(U)$, and we have a long exact sequence

$$0 \longrightarrow \mathrm{H}^1(C) \longrightarrow \mathrm{H}^1(U) \xrightarrow{\partial_z} \bigoplus_{z\in\mathcal{F}} \mathrm{H}^0(\kappa(z), -1) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{H}^q(C) \longrightarrow \mathrm{H}^q(U) \xrightarrow{\partial_z} \bigoplus_{z\in\mathcal{F}} \mathrm{H}^{q-1}(\kappa(z), -1) \longrightarrow \cdots \tag{4.7}$$

As $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}) = \varinjlim_U \mathrm{H}^1(U)$, where the limit is over all such U, $\mathrm{H}^1(C) \to \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}})$ is injective by the exactness of the injective limit functor, proving (a).

For (b) we go back to $\Lambda = \mathbb{Z}/n$. By (4.3) we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \xrightarrow{\phi_1} \mathrm{H}^0(\kappa(C), \mathbb{Z}/n) \xrightarrow{\phi_2} \mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \xrightarrow{\phi_3}$$
$$\xrightarrow{\phi_3} \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \xrightarrow{\phi_4} \mathrm{H}^1(\kappa(C), \mathbb{Z}/n)$$

We compute $\mathrm{H}^0(\kappa(C), \mathbb{Z}/n) = \bigoplus_{i=1}^m \mathrm{H}^0(\kappa(C_i), \mathbb{Z}/n) = (\mathbb{Z}/n)^m$, and $\mathrm{H}^0(C, \mathbb{Z}/n) = (\mathbb{Z}/n)^N$ where N is the number of C's connected components. Since $\mathrm{H}^0(C, \mathbb{Z}/n) = \mathrm{H}^0(U, \mathbb{Z}/n)$ for all U containing $\mathcal{S}$, and $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ is the direct limit of the $\mathrm{H}^q(U, \mathbb{Z}/n)$, we have $\mathrm{H}^0(C, \mathbb{Z}/n) = \mathrm{H}^0(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) = (\mathbb{Z}/n)^N$ by the exactness of the direct limit functor.

We claim $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ is a finite free $\mathbb{Z}/n$-module. For by (4.5), for each $z \in \mathcal{S}$ we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0(\kappa(z), \mathbb{Z}/n) \longrightarrow \mathrm{H}^0(K_{i,z}, \mathbb{Z}/n) \oplus \mathrm{H}^0(K_{j,z}, \mathbb{Z}/n) \longrightarrow \mathrm{H}^1_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mathbb{Z}/n) \to 0$$

This shows $\mathrm{H}^1_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mathbb{Z}/n) \simeq \mathbb{Z}/n$, and since $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ is a finite direct sum of these groups, it is a finite free $\mathbb{Z}/n$-module, of rank $|\mathcal{S}| = E$.

The result [14, 27.1] implies that a free $\mathbb{Z}/n$-submodule of a $\mathbb{Z}/n$-module is a direct summand. Therefore we have a decomposition

$$\mathrm{H}^0(\kappa(C), \mathbb{Z}/n) \simeq \operatorname{im}(\phi_1) \oplus \operatorname{im}(\phi_2)$$

and since $\mathrm{H}^0(\kappa(C), \mathbb{Z}/n)$ is a finite free $\mathbb{Z}/n$-module, $\operatorname{im}(\phi_2)$ is a finite free $\mathbb{Z}/n$-module by the structure theorem for finitely generated abelian groups. Similarly, since $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ is a finite free $\mathbb{Z}/n$-module,

$$\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \simeq \operatorname{im}(\phi_2) \oplus \operatorname{cok}(\phi_2)$$

and since $\mathrm{im}(\phi_2)$ and $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ are finite free $\mathbb{Z}/n$-modules, so is $\mathrm{cok}(\phi_2)$. Since $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ is a $\mathbb{Z}/n$-module, $\mathrm{cok}(\phi_2)$ is a direct summand of $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$, again by [14, 27.1]. Thus we have a decomposition

$$\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \simeq \mathrm{cok}(\phi_2) \oplus \mathrm{im}(\phi_4)$$

Now we set $\Gamma = \mathrm{im}(\phi_4)$, and compute $\mathrm{rk}(\mathrm{cok}(\phi_2)) = N + |\mathcal{S}| - m = N + E - V = \beta_C$. This proves the $q = 1$ part of (b).

The $q = 0$ case of (b) is in the proof of Lemma 4.2. Suppose $q = 2$. To show $\mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}, \mu_n) \to \mathrm{H}^2(\kappa(C), \mu_n)$ is injective, we will show $\mathrm{H}^1(\kappa(C), \mu_n) \to \mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mu_n)$ is onto and apply the exactness of (4.3).

For each closed point $z \in C_1 \cap C_2 \subset \mathcal{S}$, we have a diagram

$$\begin{array}{ccccccccc} & & & & \mathrm{H}^1(\kappa(C_1), \mu_n) \oplus \mathrm{H}^1(\kappa(C_2), \mu_n) & \longrightarrow & \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n) & & \\ & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{H}^1(\kappa(z), \mu_n) & \longrightarrow & \mathrm{H}^1(\kappa(C_1)_z, \mu_n) \oplus \mathrm{H}^1(\kappa(C_2)_z, \mu_n) & \longrightarrow & \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n) & \longrightarrow & 0 \end{array}$$

We will show that $\mathrm{H}^1(\kappa(C_1), \mu_n) \oplus \mathrm{H}^1(\kappa(C_2), \mu_n) \to \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n)$ is onto, by showing the downarrow is onto. Since z is a regular point of each C_i, each $\mathrm{O}_{C_i,z}$ is a discrete valuation ring with residue field $\kappa(z)$ and fraction field $\kappa(C_i)$, and we have a diagram of split short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1(\mathrm{O}_{C_i,z}, \mu_n) & \longrightarrow & \mathrm{H}^1(\kappa(C_i), \mu_n) & \longrightarrow & \mathrm{H}^0(\kappa(z), \mathbb{Z}/n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{H}^1(\mathrm{O}^{\mathrm{h}}_{C_i,z}, \mu_n) & \longrightarrow & \mathrm{H}^1(\kappa(C_i)_z, \mu_n) & \longrightarrow & \mathrm{H}^0(\kappa(z), \mathbb{Z}/n) & \longrightarrow & 0 \end{array}$$

To show the middle downarrow is onto it suffices (by a standard diagram chase) to prove that the left downarrow is onto. Since $\mathrm{O}^{\mathrm{h}}_{C_i,z}$ is henselian $\mathrm{H}^1(\mathrm{O}^{\mathrm{h}}_{C_i,z}, \mu_n) = \mathrm{H}^1(\kappa(z), \mu_n)$, and by Kummer theory and Hilbert 90 we have $\mathrm{H}^1(\mathrm{O}_{C_i,z}, \mu_n) = \mathrm{O}^*_{C_i,z}/n$ and $\mathrm{H}^1(\kappa(z), \mu_n) = \kappa(z)^*/n$. Since $\mathrm{O}_{C_i,z} \to \kappa(z)$ is onto and $\mathrm{O}_{C_i,z}$ is local, the induced map $\mathrm{O}^*_{C_i,z} \to \kappa(z)^*$ is onto, hence $\mathrm{H}^1(\mathrm{O}_{C_i,z}, \mu_n)$ maps onto $\mathrm{H}^1(\kappa(z), \mu_n)$. We conclude $\mathrm{H}^1(\kappa(C_i), \mu_n) \to \mathrm{H}^1(\kappa(C_i)_z, \mu_n)$ is onto. Now each map $\mathrm{H}^1(\kappa(C_1), \mu_n) \oplus \mathrm{H}^1(\kappa(C_2), \mu_n) \to \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n)$ is onto.

Suppose $(b_z)_{z\in\mathcal{S}} \in \mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mu_n) = \bigoplus_{z\in\mathcal{S}} \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n)$. We have just seen that for each closed point $z \in C_i \cap C_j$ there exists a pair $((a_{i,z}), (a_{j,z})) \in \kappa(C_i)^*/n \oplus \kappa(C_j)^*/n$ mapping to b_z, for elements $a_{k,z} \in \mathrm{O}_{C_k,z}$, for $k = i, j$. Let $v_{k,z}$ be the discrete valuation on $\kappa(C_k)$ determined by z. By standard approximation (e.g. [29, XII.1.2]) there exist elements $a_k \in \kappa(C_k)$ such that $v_{k,z}(a_k - a_{k,z}) > v_{k,z}(a_{k,z})$ for all z. The image of a_k in $\kappa(C_k)^*_z/n$ is then $(a_{k,z})$, since the group $U^{(1)}_z = \{u \in \kappa(C_k)^*_z : v_{k,z}(u-1) > 0\}$ is an n-divisible subgroup of the unit group of the henselian field $\kappa(C_k)_z$. Therefore the m-tuple $((a_k))_k \in \bigoplus_{k=1}^m \mathrm{H}^1(\kappa(C_i), \mu_n) = \mathrm{H}^1(\kappa(C), \mu_n)$ maps to (b_z). This proves the induced map $\mathrm{H}^1(\kappa(C), \mu_n) \to \mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mu_n)$ is onto, and completes the proof. □

We will soon need the following technical lemma in order to replace X_0 with C.

Lemma 4.8. *Suppose A is a noetherian ring. Then the natural map* $(\mathrm{Frac}\, A)_{\mathrm{red}} \to \mathrm{Frac}\,(A_{\mathrm{red}})$ *is an isomorphism if and only if A has no embedded primes.*

Proof. Let $f : A \to A_{\mathrm{red}} = A/N_A$ be the natural map, where N_A is the nilradical of A, let $S = A - \bigcup_{\mathrm{Ass}A} \mathfrak{p}$, where $\mathrm{Ass}A$ is the set of associated primes of A, and let $T = A - \bigcup_{\mathrm{Min}A} \mathfrak{p}$, where $\mathrm{Min}A$ is the set of minimal primes of A. Then S is the set of non zero-divisors of A, $S \subset T$, and since $\mathrm{Min}(A_{\mathrm{red}}) = \mathrm{Ass}(A_{\mathrm{red}})$, $f(T)$ is the set of non zero-divisors of A_{red} (see e.g. [4, Chapter 4]).

We have $(\mathrm{Frac}\, A)_{\mathrm{red}} = (S^{-1}A)_{\mathrm{red}} = S^{-1}A/N_{S^{-1}A}$ by definition, and the latter is $S^{-1}A/S^{-1}N_A = f(S)^{-1}A_{\mathrm{red}}$ by [4, 3.12]. Since $\mathrm{Frac}\,(A_{\mathrm{red}}) = f(T)^{-1}A_{\mathrm{red}}$ and $f(S) \subset f(T)$, we have a natural map

$$(\mathrm{Frac}\, A)_{\mathrm{red}} = f(S)^{-1}A_{\mathrm{red}} \longrightarrow f(T)^{-1}A_{\mathrm{red}} = \mathrm{Frac}\,(A_{\mathrm{red}})$$

By [4, Exercise 3.8(v)] this is an isomorphism if and only if every prime ideal of A_{red} that meets $f(T)$ also meets $f(S)$. Since every prime of A is the preimage of its image under f, the latter condition is equivalent to: Every prime of A that meets T also meets S. Every prime that meets T also meets S if and only if $\mathrm{Ass}A \subset \bigcup_{\mathrm{Min}A} \mathfrak{p}$, if and only if $\mathrm{Ass}A \subset \mathrm{Min}A$, by prime avoidance ([4, 1.11]). Since always $\mathrm{Ass}A \supset \mathrm{Min}A$, we conclude that we have an isomorphism $(\mathrm{Frac}\, A)_{\mathrm{red}} \xrightarrow{\sim} \mathrm{Frac}\,(A_{\mathrm{red}})$ if and only if $\mathrm{Min}A = \mathrm{Ass}A$, i.e., A has no embedded primes. □

Theorem 4.9. *Assume the setup of (2.4). Then for $q \geq 0$ there is a map $\lambda : \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \Lambda) \to \mathrm{H}^q(F, \Lambda)$ and a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \Lambda) & \xrightarrow{\lambda} & \mathrm{H}^q(F, \Lambda) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{H}^q(\kappa(C), \Lambda) & \xrightarrow{\mathrm{inf}} & \mathrm{H}^q(F_C, \Lambda) \end{array} \tag{4.10}$$

such that if $\alpha_C \in \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \Lambda)$ and $\alpha = \lambda(\alpha_C)$ then:

a) *α is defined at the generic point of each C_i, and $\alpha(C_i) = \alpha_{\kappa(C_i)} = (\alpha_C)_{\kappa(C_i)}$.*
b) *The ramification locus of α (on X) is contained in $\mathscr{D}_{\mathcal{S}}$.*
c) *If $D \in \mathscr{D}_{\mathcal{S}}$ is prime and $z = D \cap C$, then $\partial_D \cdot \lambda = \inf_{\kappa(z)|\kappa(D)} \cdot \partial_z$.*
d) *If α_C is unramified at a closed point z, and D is any (horizontal) prime lying over z, then α is unramified at D, and has value $\alpha(D) = \inf_{\kappa(z)|\kappa(D)}(\alpha_C(z))$.*

Proof. Let D_C be an effective divisor on C that avoids $\mathcal{S}$, let $D \in \mathscr{D}_{\mathcal{S}}$ be the distinguished lift of D_C, set $U = X - D$, and set $U_C = C - D_C$. We will say such U are *distinguished*. Since X and D are regular and D has pure codimension 1, we have $\mathrm{H}^0(X) \simeq \mathrm{H}^0(U)$, and an exact Gysin sequence

$$0 \longrightarrow \mathrm{H}^1(X) \longrightarrow \mathrm{H}^1(U) \xrightarrow{\partial_D} \mathrm{H}^0(D, -1) \longrightarrow \mathrm{H}^2(X) \longrightarrow \cdots$$

by Gabber's absolute purity theorem ([15, Theorem 2.1.1]) and the standard construction of the Gysin sequence ([12, Section 3.2]). (Note that the result in [15] is stated for the $\Lambda = \mathbb{Z}/n$ case only, but the result holds in general since the sheaves $\mathscr{H}^q_D(X)$ and $\mathscr{H}^q_D(X, \mathbb{Z}/n)$ are locally isomorphic, and the morphism $i^*\Lambda(-1) \to \mathscr{H}^2_D(X)$ is canonical, where $i : D \to X$ stands for the closed immersion.) We use the notation ∂_D since this map is compatible with the one defined above on $\mathrm{H}^q(F)$ when D is prime.

We may replace X_0 by $C = X_{0,\mathrm{red}}$ in the cohomological computations below since Λ is finite and n is prime-to-p, by [33, V.2.4(c)] (see also [33, II.3.11]). To substitute $\mathrm{O}_{C,\mathcal{S}}$ and $\kappa(C)$ for $\mathrm{O}_{X_0,\mathcal{S}}$ and $\kappa(X_0)$ we must check that the former are the canonical reduced quotients of the latter. But the ring $\mathrm{O}_{X_0,\mathcal{S}}$ can by obtained by localizing some affine open subset Spec A_0 containing $\mathcal{S}$ (which exists since X_0/k is projective) with respect to the multiplicative set $T = A_0 - \bigcup_{z\in\mathcal{S}} \mathfrak{m}_z$. Since $\mathrm{O}_{C,\mathcal{S}}$ is obtained by localizing $A_{0,\mathrm{red}}$ with respect to the image of T in $A_{0,\mathrm{red}}$, we have $\mathrm{O}_{C,\mathcal{S}} = (\mathrm{O}_{X_0,\mathcal{S}})_{\mathrm{red}}$ since the formation of the nilradical commutes with localization (see e.g. [4, 3.12]).

To show $\kappa(C) = \kappa(X_0)_{\mathrm{red}}$ it suffices to show X_0 has no embedded points by Lemma 4.8. But if z is any closed point of X then $\mathrm{O}_{X,z}$ is a regular local ring, and a local equation for the closed fiber $\mathrm{Spec}\,(\mathrm{O}_{X,z} \otimes_R k)$ passing through z is given by the uniformizer p in R. Since $\mathrm{O}_{X,z}$ is factorial and at most two components of X_0 pass through z we have $p = u\pi_1^{e_1}\pi_2^{e_2}$ for $u \in \mathrm{O}_{X,z}^*$, primes π_i, and numbers $e_i \geq 0$. The associated primes of $\mathrm{O}_{X,z}/(u\pi_1^{e_1}\pi_2^{e_2})$ are evidently just the (π_i), which shows X_0 has no embedded point at z.

Since D_C is a disjoint union of regular closed points, by (4.7) and the work that immediately precedes it we have $\mathrm{H}^0(C) \simeq \mathrm{H}^0(U_C)$ and an exact sequence

$$0 \longrightarrow \mathrm{H}^1(C) \longrightarrow \mathrm{H}^1(U_C) \xrightarrow{\partial_{D_C}} \mathrm{H}^0(\kappa(D_C), -1) \longrightarrow \mathrm{H}^2(C) \longrightarrow \cdots$$

Thus we have a commutative ladder

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{H}^1(X) & \longrightarrow & \mathrm{H}^1(U) & \xrightarrow{\partial_D} & \mathrm{H}^0(D,-1) & \longrightarrow & \mathrm{H}^2(X) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{H}^1(C) & \longrightarrow & \mathrm{H}^1(U_C) & \xrightarrow{\partial_{D_C}} & \mathrm{H}^0(D_C,-1) & \longrightarrow & \mathrm{H}^2(C) & \longrightarrow \cdots
\end{array}$$

Since R is complete, $\mathrm{H}^q(X) \to \mathrm{H}^q(C)$ and $\mathrm{H}^q(D,-1) \to \mathrm{H}^q(D_C,-1)$ are isomorphisms for $q \geq 0$ by proper base change ([33, VI.Corollary 2.7]). Therefore, in light of the isomorphisms in degree zero and the 5-lemma in degree $q \geq 1$, we obtain isomorphisms

$$\mathrm{H}^q(U) \xrightarrow{\sim} \mathrm{H}^q(U_C)$$

for $q \geq 0$. Taking the limit over all $U_C = C - D_C$ (containing $\mathcal{S}$) and corresponding $U = X - D$ for $D \in \mathscr{D}_{\mathcal{S}}$ lying over D_C yields an isomorphism

$$\varinjlim \mathrm{H}^q(U) \xrightarrow{\sim} \varinjlim \mathrm{H}^q(U_C) = \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}) \tag{4.11}$$

On the other hand, for each distinguished U we have a map $\mathrm{H}^q(U) \to \mathrm{H}^q(F)$, hence a map $\varinjlim \mathrm{H}^q(U) \to \mathrm{H}^q(F)$. Composing with the inverse of (4.11) yields our lift

$$\lambda : \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^q(F)$$

Applying cohomology to the commutative diagram

$$\begin{array}{ccccc}
U_C & \longrightarrow & U & \longleftarrow & \mathrm{Spec}\, F \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Spec}\,\kappa(C_i) & \longrightarrow & \mathrm{Spec}\,\mathrm{O}_{F_{C_i}} & \longleftarrow & \mathrm{Spec}\, F_{C_i}
\end{array}$$

yields a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^q(U_C) & = & \mathrm{H}^q(U) & \longrightarrow & \mathrm{H}^q(F) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^q(\kappa(C_i)) & = & \mathrm{H}^q(\mathrm{O}_{F_{C_i}}) & \longrightarrow & \mathrm{H}^q(F_{C_i}) \end{array}$$

Taking the limit again over U_C and U yields the diagram (4.10). If $\alpha_C \in \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}})$ and $\alpha = \lambda(\alpha_C)$ then since each U contains $\mathcal{S}$ and the generic points of the C_i, α is defined at these points, and the formula $\alpha(C_i) = \mathrm{res}_{\mathrm{O}_{C,\mathcal{S}}|\kappa(C_i)}(\alpha_C)$ follows immediately from (4.10). Since the restriction map $\mathrm{H}^q(\mathrm{O}_{X,\kappa(C_i)}) \to \mathrm{H}^q(\kappa(C_i))$ factors through $\mathrm{H}^q(\mathrm{O}_{F_{C_i}})$, we have $\alpha(C_i) = \alpha_{\kappa(C_i)}$. This proves (a).

If D is a horizontal prime divisor not in $\mathscr{D}_{\mathcal{S}}$, then the generic point $\operatorname{Spec} \kappa(D)$ is contained in each distinguished U, hence $\varinjlim \mathrm{H}^q(U) \to \mathrm{H}^q(F_D)$ factors through $\mathrm{H}^q(\mathrm{O}_{F_D})$, which shows $\partial_D \cdot \lambda = 0$. Thus the ramification locus of any element in the image of λ must be contained in $\mathscr{D}_{\mathcal{S}}$, proving (b). Now if $D \in \mathscr{D}_{\mathcal{S}}$ is prime and $z = D \cap C$ then D is the prime spectrum of a complete local ring with residue field $\kappa(z)$, and the isomorphism

$$\mathrm{H}^{q-1}(D, -1) \xrightarrow{\sim} \mathrm{H}^{q-1}(z, -1) = \mathrm{H}^{q-1}(\kappa(z), -1)$$

is the standard identification. Thus the formula $\partial_D \cdot \lambda = \inf_{\kappa(z)|\kappa(D)} \cdot \partial_z$ is immediate by the commutative ladder of Gysin sequences above and the compatibility of ∂_D with the residue map on $\mathrm{H}^q(F)$, proving (c).

Suppose $\alpha = \lambda(\alpha_C)$ has ramification locus D_α, then $D_\alpha \in \mathscr{D}_{\mathcal{S}}$. Set $U = X - D_\alpha$ and $U_C = U \cap C = C - D_{\alpha_C}$. If α_C is unramified at a point z, i.e., $z \in U_C$, then α is unramified at every prime divisor D lying over z. For if $D \in \mathscr{D}_{\mathcal{S}}$ then $\partial_D(\alpha) = \inf(\partial_z(\alpha_C))$ by the formula just proved, and if $D \notin \mathscr{D}_{\mathcal{S}}$ then $\partial_D(\alpha) = 0$ since $D_\alpha \in \mathscr{D}_{\mathcal{S}}$. Thus if α_C is unramified at z, and D is a prime divisor lying over z, then U contains z and $\operatorname{Spec} \kappa(D)$, hence $U \supset D$. The maps $z = \operatorname{Spec} \kappa(z) \to U_C$ and $D \to U$ then induce a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^q(U) & \xrightarrow{\mathrm{res}} & \mathrm{H}^q(D) & \xrightarrow{\mathrm{res}} & \mathrm{H}^q(\kappa(D)) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & \nearrow{\scriptstyle \mathrm{inf}} & \\ \mathrm{H}^q(U_C) & \xrightarrow{\mathrm{res}} & \mathrm{H}^q(z) & & \end{array}$$

Both vertical down-arrows are isomorphisms by the commutative ladder. The inverse of the left one is λ by definition, and the composition of the inverse of the right one and the restriction $\mathrm{H}^q(D) \to \mathrm{H}^q(\kappa(D))$ is inflation, as shown. The top composition of horizontal restrictions factors through the restriction $\mathrm{H}^q(U) \to \mathrm{H}^q(\widehat{\mathrm{O}}_{X,D})$ and the bottom factors through the restriction $\mathrm{H}^q(U_C) \to \mathrm{H}^q(\widehat{\mathrm{O}}_{C,z})$. Since these are restriction maps, the images of α and α_C are the values $\alpha(D)$ and $\alpha_C(z)$. We conclude $\inf_{\kappa(z)|\kappa(D)}(\alpha_C(z)) = \alpha(D)$, as in (d). □

Lemma 4.12. *Assume the setup of* (2.4)*, let $D \in \mathscr{D}_{\mathcal{S}}$, and suppose $\rho : Y \to (X, D)$ is a tamely ramified cover. Then there is a map $\lambda_Y : \mathrm{H}^q(\mathrm{O}_{C_Y,\mathcal{S}_Y}, \Lambda) \to \mathrm{H}^q(K(Y), \Lambda)$*

inducing a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^q(\mathrm{O}_{C_Y,\mathcal{S}_Y},\Lambda) & \xrightarrow{\lambda_Y} & \mathrm{H}^q(K(Y),\Lambda) \\
{\scriptstyle \mathrm{res}}\uparrow & & \uparrow{\scriptstyle \mathrm{res}} \\
\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda) & \xrightarrow[\lambda]{} & \mathrm{H}^q(F,\Lambda)
\end{array}$$

Proof. Since $\rho : Y \to (X, D)$ is a tamely ramified cover, Y/R is a regular relative curve, each irreducible component of $C_Y = (Y_0)_{\mathrm{red}}$ is regular, and exactly two irreducible components of C_Y meet at each singular point $z \in \mathcal{S}_Y$, all by Lemma 3.1. Thus Y/R satisfies the hypotheses of (2.4), and we have the map λ_Y by Theorem 4.9, which is defined relative to the set of distinguished divisors $\mathscr{D}_Y$ as in Lemma 3.1(c).

Let $U_C \subset C$ be any open set containing $\mathcal{S}$, set $D'_C = C - U_C$, and let $D' \in \mathscr{D}_{\mathcal{S}}$ be the distinguished divisor lying over D'_C. Then let $U = X - D'$, $V = Y \times_X U$, and $V_C = U_C \times_U V$. Then we have a commutative diagram

$$\begin{array}{ccc}
V_C & \longrightarrow & V \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
U_C & \longrightarrow & U
\end{array}$$

Since $\mathcal{S}_Y = \rho^{-1}\mathcal{S}$, V and V_C contain $\mathcal{S}_Y$, and applying cohomology yields a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^q(\mathrm{O}_{C_Y,\mathcal{S}_Y}) & = & \varinjlim \mathrm{H}^q(V'_C) & \xrightarrow{\sim} & \varinjlim \mathrm{H}^q(V') & \searrow & \\
\uparrow & & {\scriptstyle \mathrm{can}}\uparrow & & {\scriptstyle \mathrm{can}}\uparrow & & \\
{\scriptstyle \mathrm{res}} & & \varinjlim \mathrm{H}^q(V_C) & \xleftarrow{\sim} & \varinjlim \mathrm{H}^q(V) & \longrightarrow & \mathrm{H}^q(K(Y)) \\
\uparrow & & {\scriptstyle \mathrm{res}}\uparrow & & {\scriptstyle \mathrm{res}}\uparrow & & \uparrow{\scriptstyle \mathrm{res}} \\
\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}) & = & \varinjlim \mathrm{H}^q(U_C) & \xleftarrow{\sim} & \varinjlim \mathrm{H}^q(U) & \longrightarrow & \mathrm{H}^q(F)
\end{array}$$

where V' runs over all open subsets of Y containing $\mathcal{S}_Y$, and $V'_C = V' \times_X C$. This yields the diagram of the lemma. $\square$

4.13. Since X is noetherian each $f \in F^*$ defines a divisor $\operatorname{div} f = \sum v_D(f)D$, where the (finite) sum is over prime divisors on X. By weak approximation [40, Lemma] there exists a $\pi \in F$ such that

$$\operatorname{div} \pi = C + H$$

where H is horizontal and avoids any finite (preassigned) set of points $\mathcal{F}$. We fix such a π for $\mathcal{F}$ containing $\mathcal{S}$. Since $v_{C_i}(\pi) = 1$ for each i, the choice of π determines a noncanonical Witt decomposition

$$\mathrm{H}^q(\kappa(C), \Lambda) \oplus \mathrm{H}^{q-1}(\kappa(C), \Lambda(-1)) \xrightarrow{\sim} \mathrm{H}^q(F_C, \Lambda)$$

taking (α_C, θ_C) to $\alpha_C + (\pi) \cdot \theta_C$, where α_C and θ_C are inflated from $\kappa(C)$ to F_C, (π) is the image of π in $\mathrm{H}^1(F_C, \mu_n)$, and $(\pi) \cdot \theta$ is the cup product. Composing this map with the natural maps $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda)\to\mathrm{H}^q(\kappa(C),\Lambda)$ and $\mathrm{H}^{q-1}(\mathrm{O}_{C,\mathcal{S}},\Lambda(-1))\to\mathrm{H}^{q-1}(\kappa(C),\Lambda(-1))$ yields a homomorphism

$$\eta_\pi:\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda)\oplus\mathrm{H}^{q-1}(\mathrm{O}_{C,\mathcal{S}},\Lambda(-1))\longrightarrow\mathrm{H}^q(F_C,\Lambda)$$

defined by $\eta_\pi(\alpha_C,\theta_C)=\alpha_C+(\pi)\cdot\theta_C$.

Corollary 4.14. *The choice of $\mathscr{D}_{\mathcal{S}}$ and π determines for $q\ge 1$ a homomorphism*

$$\lambda:\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda)\oplus\mathrm{H}^{q-1}(\mathrm{O}_{C,\mathcal{S}},\Lambda(-1))\longrightarrow\ \mathrm{H}^q(F,\Lambda)$$

by the rule $\lambda(\alpha_C+(\pi)\cdot\theta_C)=\lambda(\alpha_C)+(\pi)\cdot\lambda(\theta_C)$, resulting in a commutative diagram

$$\begin{array}{ccc} & & \mathrm{H}^q(F,\Lambda)\\ & \nearrow^{\lambda} & \Big\downarrow{\scriptstyle \mathrm{res}_{F|F_C}}\\ \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda)\oplus\mathrm{H}^{q-1}(\mathrm{O}_{C,\mathcal{S}},\Lambda(-1)) & \xrightarrow[\eta_\pi]{} & \mathrm{H}^q(F_C,\Lambda)\end{array}$$

If $D\in\mathscr{D}_{\mathcal{S}}$ and $\rho:Y\to(X,D)$ is a tamely ramified cover then we have a commutative diagram

$$\begin{array}{ccccc}\mathrm{H}^q(\mathrm{O}_{C_Y,\mathcal{S}_Y},\Lambda) & \oplus & \mathrm{H}^{q-1}(\mathrm{O}_{C_Y,\mathcal{S}_Y},\Lambda(-1)) & \xrightarrow{\lambda_Y} & \mathrm{H}^q(K(Y),\Lambda)\\ \Big\uparrow{\scriptstyle\mathrm{res}} & & \Big\uparrow{\scriptstyle\mathrm{res}} & & \Big\uparrow{\scriptstyle\mathrm{res}}\\ \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda) & \oplus & \mathrm{H}^{q-1}(\mathrm{O}_{C,\mathcal{S}},\Lambda(-1)) & \xrightarrow[\lambda]{} & \mathrm{H}^q(F,\Lambda)\end{array}$$

where λ_Y is defined using $\mathscr{D}_{\mathcal{S}_Y}$, π, and Lemma 4.12.

Proof. This is an immediate consequence of Theorem 4.9 and Lemma 4.12. □

Remark 4.15. If X/R is smooth, then $\mathcal{S}$ is empty, and $\mathrm{O}_{C,\mathcal{S}}=\kappa(C)$. The map η_π is then an isomorphism, and incorporating it into λ we obtain a map

$$\lambda:\mathrm{H}^q(F_C,\Lambda)\longrightarrow\mathrm{H}^q(F,\Lambda)$$

that splits the restriction map. This is the map of [10]. Note however that Corollary 4.14 does not imply a well defined map $\mathrm{im}(\eta_\pi)\to\mathrm{H}^q(F,\Lambda)$. In fact there is an obstruction, at least part of which lies in the prospect of nontrivial elements of $\mathrm{im}(\lambda)\le\mathrm{H}^q(F,\Lambda)$ that are trivial at $\kappa(C)$. We discuss the $q=1$ case in (4.16).

4.16. Completely Split Elements. In [36, 2.1] Saito defines a completely split covering of a noetherian scheme X to be a finite étale cover $Y\to X$ such that $Y\times_X\operatorname{Spec}\kappa(x)=\coprod\operatorname{Spec}\kappa(x)$, for all closed points $x\in X$. We abuse Saito's terminology (see Remark(4.18) below) and in the setup of (2.4) denote by $\mathrm{H}^1_{\mathrm{cs}}(C,\Lambda)$ the kernel of the map $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\Lambda)\to\mathrm{H}^1(\kappa(C),\Lambda)$, so that we have an exact sequence

$$0\longrightarrow\mathrm{H}^1_{\mathrm{cs}}(C,\Lambda)\longrightarrow\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\Lambda)\longrightarrow\mathrm{H}^1(\kappa(C),\Lambda)$$

Then $\mathrm{H}^1_{\mathrm{cs}}(C,\Lambda)\le\mathrm{H}^1(C,\Lambda)$: For if $\beta\in\mathrm{H}^1_{\mathrm{cs}}(C,\Lambda)$ then $\partial_z(\beta)=0$ for all closed points $z\in C-\mathcal{S}$ since ∂_z factors through $\kappa(C_i)_z$. Therefore β is defined on C, and since $\mathrm{H}^1(C,\Lambda)\le\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\Lambda)$ by Lemma 4.6(a), $\mathrm{H}^1_{\mathrm{cs}}(C,\Lambda)\le\mathrm{H}^1(C,\Lambda)$, as claimed. Let $\mathrm{H}^1_{\mathrm{cs}}(X,\Lambda)$ denote the preimage of $\mathrm{H}^1_{\mathrm{cs}}(C,\Lambda)$ under the proper base change isomorphism $\mathrm{H}^1(X,\Lambda)\xrightarrow{\sim}\mathrm{H}^1(C,\Lambda)$.

Proposition 4.17. *Assume the setup of (2.4). Then elements of* $\mathrm{H}^1_{\mathrm{cs}}(C,\Lambda)$ *are trivial at all points of* C*, and the nontrivial elements of* $\mathrm{H}^1_{\mathrm{cs}}(X,\Lambda)$ *are trivial at all points of* X *except for the generic point* $\operatorname{Spec} F$*, where they are nontrivial.*

Proof. Suppose $\beta_C \in \mathrm{H}^1_{\mathrm{cs}}(C)$. Then β_C is trivial at each generic point of C by definition of $\mathrm{H}^1_{\mathrm{cs}}(C)$. If $z \in C$ is a closed point lying on the irreducible component C_i then the map $\mathrm{H}^1(C) \to \mathrm{H}^1(\kappa(z))$ factors through $\mathrm{H}^1(C_i)$. Since C_i is regular the map $\mathrm{H}^1(C_i) \to \mathrm{H}^1(\kappa(C_i))$ is injective by purity, and consequently $\beta_C(z) = 0$ by definition. Thus the elements of $\mathrm{H}^1_{\mathrm{cs}}(C)$ are trivial at all points of C.

Suppose $\beta = \lambda(\beta_C) \in \mathrm{H}^1_{\mathrm{cs}}(X)$. If $x \in X$ is a generic point of some irreducible component C_i of C then the image of β in $\mathrm{H}^1(\kappa(C_i))$ is zero since the map $\mathrm{H}^1_{\mathrm{cs}}(X) \to \mathrm{H}^1(\kappa(C_i))$ factors through $\mathrm{H}^1_{\mathrm{cs}}(C)$. If x is the generic point of a horizontal divisor D with closed point z then $\beta(D) = \inf_{\kappa(z)|\kappa(D)}(\beta_C(z))$ by Theorem 4.9(d), and this is zero since $\beta_C(z) = 0$. If z is a closed point of X then z is on C, and the map $\mathrm{H}^1_{\mathrm{cs}}(X) \to \mathrm{H}^1(\kappa(z))$ factors through $\mathrm{H}^1_{\mathrm{cs}}(C)$, hence β is trivial at z. Finally, since X is a regular noetherian scheme the map $\mathrm{H}^1(X) \to \mathrm{H}^1(F)$ is injective by purity, hence β is nontrivial at the generic point of X. ☐

Remark 4.18. Proposition 4.17 shows the elements of $\mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)$ are completely split in the sense of [36], since they are split at all closed points. However, $\mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)$ does not contain elements that are split at all closed points but nontrivial at generic points of C. Such elements do not exist if k is finite, as shown by Saito in [36, Theorem 2.4], since then the C_i have no nontrivial completely split covers, essentially by Cebotarev's density theorem (see [35, Lemma 1.7]).

5. Applications in the Brauer Group

5.1. Cyclic Covers. If U is any scheme, and $\bar{u}$ is a geometric point, the fiber functor defines a category equivalence between (finite) étale covers of U and finite continuous $\pi_1(U,\bar{u})$-sets, yielding a canonical isomorphism

$$\mathrm{H}^1(U,\mathbb{Z}/n) \simeq \mathrm{H}^1(\pi_1(U,\bar{u}),\mathbb{Z}/n) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U,\bar{u}),\mathbb{Z}/n) \tag{5.2}$$

(see [13, I.2.11]). If $\theta \in \mathrm{H}^1(U,\mathbb{Z}/n)$, we will write $U[\theta]$ for the finite cyclic étale cover determined by θ. If $U = \operatorname{Spec} A$ is affine, we will write $A[\theta]$ for the corresponding ring, or $A(\theta)$ if A is a field. If U is a connected normal scheme, and $\theta \in \mathrm{H}^1(U,\mathbb{Z}/n)$ has order m, then $U[\theta]$ is a disjoint sum of n/m connected $\mathbb{Z}/m$-Galois covers of U.

If $V \to U$ is a morphism and $\bar{v}$ is a geometric point of V (hence of U), base extension defines a homomorphism $\pi_1(V,\bar{v}) \to \pi_1(U,\bar{v})$, inducing the restriction map $\mathrm{H}^1(U,\mathbb{Z}/n) \to \mathrm{H}^1(V,\mathbb{Z}/n)$ via (5.2). If $\theta \in \mathrm{H}^1(U,\mathbb{Z}/n)$ maps to $\theta_V \in \mathrm{H}^1(V,\mathbb{Z}/n)$ then $U[\theta] \times_U V = V[\theta_V]$.

In the setup of (2.4), if $U \subset X$ is an open subscheme then since X is excellent and regular, the map $\mathrm{H}^1(U,\mathbb{Z}/n) \to \mathrm{H}^1(F,\mathbb{Z}/n)$ is injective (see e.g. the proof of [12, Corollary 3.4.2]). Thus if $\theta \in \mathrm{H}^1(F,\mathbb{Z}/n)$ and θ is defined on $U \subset X$ then we may view θ as an element of $\mathrm{H}^1(U,\mathbb{Z}/n)$. Then $U[\theta] \times_U \operatorname{Spec} F = \operatorname{Spec} F(\theta)$, and since $U[\theta] \to U$ is étale, $F(\theta)$ is the total fraction ring of $U[\theta]$, and $U[\theta]$ is the normalization of U in $\operatorname{Spec} F(\theta)$.

Lemma 5.3. *Assume the setup of (2.4). Suppose* $\theta_C \in \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ *has ramification divisor* D_C *on* C*,* $\theta = \lambda(\theta_C)$*,* $D \in \mathscr{D}_{\mathcal{S}}$ *is the distinguished lift of* D_C*, and* Y *is the normalization of* X *in* $F(\theta)$*. Then* $\rho : Y \to (X, D)$ *is a tamely ramified cover whose restriction to* C *is a tamely ramified cover* $\rho_C : C_Y \to (C, D_C)$*, and* $\mathrm{O}_{C_Y,\mathcal{S}_Y} = \mathrm{O}_{C,\mathcal{S}}[\theta_C]$.

Proof. The lift θ is tamely ramified with respect to D by Theorem 4.9. If $\theta \in \mathrm{H}^1(F, \mathbb{Z}/n)$, v is any valuation on F, and w is an extension of v to $F(\theta)$, then the ramification index $e(w/v)$ equals $|\partial_v(\theta)|$, by e.g. [16, Chapter II]. Therefore, since X is regular, $\rho : Y \to (X, D)$ is a tamely ramified cover by Lemma 3.2.

Let $U = X - D$, $V = U \times_X Y$, $U_C = U \times_X C \subset C$ and $V_C = V \times_U U_C \subset C_Y$. The construction of the map λ in (4.11) shows that θ_C and θ may be viewed as elements of $\mathrm{H}^1(U_C, \mathbb{Z}/n)$ and $\mathrm{H}^1(U, \mathbb{Z}/n)$, respectively, and $\mathrm{res}_{U|U_C}(\theta) = \theta_C$. Since Y is the normalization of X in $F(\theta)$, V is the normalization of U in $F(\theta)$, hence $V = U[\theta]$. Since $\mathrm{res}_{U|U_C} : \mathrm{H}^1(U, \mathbb{Z}/n) \to \mathrm{H}^1(U_C, \mathbb{Z}/n)$ is induced on covers by base change, we conclude $V_C = V \times_U U_C = U_C[\theta_C]$. Now if $U'_C \subset U_C$ is any open set containing $\mathcal{S}$ then we may view θ_C as an element of $\mathrm{H}^1(U'_C, \mathbb{Z}/n)$, and $V'_C = V_C \times_{U_C} U'_C = U'_C[\theta_C]$. Taking the limit over all such U'_C yields the ring $\mathrm{O}_{C,\mathcal{S}}[\theta_C]$, and since $\mathcal{S}_Y = \rho^{-1}\mathcal{S}$, this is $\mathrm{O}_{C_Y,\mathcal{S}_Y}$ by [4, Exercise 3.8] (observe that $\operatorname{Spec} \mathrm{O}_{C_Y,\mathcal{S}_Y} = \operatorname{Spec} \mathrm{O}_{C,\mathcal{S}} \times_C C_Y$).

The map $C_Y \to C$ induced by $Y \to X$ is finite and étale over $U_C = C - D_C$ by base change. Each irreducible component of C_Y is regular by Lemma 3.1, and dominates an irreducible component of C, hence $C_Y \to (C, D_C)$ is a totally ramified cover by the definition in (2.3). This completes the proof. □

5.4. Index. Let $\pi \in F$ and η_π be as in (4.13), with $q = 2$. Let $\delta_C = \alpha_C + (\pi) \cdot \theta_C$ be in $\mathrm{H}^2(F_C, \mu_n)$, and let $\delta_{C_i} = \alpha_{C_i} + (\pi) \cdot \theta_{C_i}$ be the image of δ_C in $\mathrm{H}^2(F_{C_i}, \mu_n)$. By the (well-known) Nakayama-Witt index formula we have $\mathrm{ind}(\delta_{C_i}) = |\theta_{C_i}| \cdot \mathrm{ind}(\alpha_{C_{i,\kappa(C_i)(\theta_{C_i})}})$. Define

$$\mathrm{ind}(\alpha_C, \theta_C) = \mathrm{lcm}_i\{\mathrm{ind}(\delta_{C_i})\}$$

Theorem 5.5. *Assume the setup of* (2.4) *with* $R = \mathbb{Z}_p$*, and let* λ *be the map of Corollary 4.14. Then* $\mathrm{ind}(\lambda(\alpha_C, \theta_C)) = \mathrm{ind}(\alpha_C, \theta_C)$.

Proof. We adopt the notation of (5.4), and set $\delta = \lambda(\alpha_C, \theta_C)$, $\alpha = \lambda(\alpha_C)$, and $\theta = \lambda(\theta_C)$, so that $\delta = \alpha + (\pi) \cdot \theta$. By primary decomposition we may assume n is a power of a prime ℓ. In Corollary 4.14 we have defined for every $X/\mathbb{Z}_p$ satisfying the setup of (2.4) a map $\lambda : \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}, \mu_n) \oplus \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2(F, \mu_n)$, which is constructed relative to π and $\mathscr{D}_{\mathcal{S}}$.

To show that λ preserves index, we proceed by induction on $\mathrm{ind}(\alpha_C, \theta_C)$. Assume first that $\mathrm{ind}(\alpha_C, \theta_C) = 1$, i.e., that all α_{C_i}, θ_{C_i} are trivial. Then $\theta_C \in \mathrm{H}^1_{\mathrm{cs}}(C, \mathbb{Z}/n)$ and $\theta \in \mathrm{H}^1_{\mathrm{cs}}(X, \mathbb{Z}/n)$ by (4.16), and $\alpha_C = 0$ by Lemma 4.6, hence $\alpha = 0$. To show that $\delta = (\pi) \cdot \theta$ is trivial we compute ramification:

$$\partial_D((\pi) \cdot \theta) = v_D(\pi) \cdot \theta_{F_D} - (\pi) \cdot \partial_D(\theta) + (-1) \cdot v_D(\pi) \cdot \partial_D(\theta)$$

Since $\theta \in \mathrm{H}^1_{\mathrm{cs}}(X, \mathbb{Z}/n)$, $\partial_D(\theta) = 0$ and $\theta_{F_D} = 0$ by Proposition 4.17, hence $\partial_D((\pi) \cdot \theta) = 0$, as desired.

Now assume $\mathrm{ind}(\alpha_C,\theta_C) = N > 1$, and that λ preserves index on elements of ℓ-power index less than N. Note this hypothesis applies to any X satisfying the setup of (2.4). We construct a tamely ramified cover $\rho : Y \to X$ of degree ℓ as follows. Set $\phi_{C_i} = (|\theta_{C_i}|/\ell)\theta_{C_i} \in \mathrm{H}^1(\kappa(C_i),\mathbb{Z}/\ell)$ if $\theta_{C_i} \neq 0$, and otherwise use Grunwald-Wang's theorem to produce an element $\phi_{C_i} \in \mathrm{H}^1(\kappa(C_i),\mathbb{Z}/\ell)$ of order ℓ such that $\phi_{C_i,z} \neq 0$ whenever $\partial_z(\alpha_{C_i}) \neq 0$, and if $z \in \mathcal{S}$ is on $C_i \cap C_j$, $\phi_{C_i,z} = \theta_{C_j,z}$. Then there exists an element $\phi_C = (\phi_{C_i})_i \in \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/\ell)$ by Lemma 4.2, and we obtain an element $\phi = \lambda(\phi_C) \in \mathrm{H}^1(F,\mathbb{Z}/\ell)$. Let $\rho : Y \to X$ be the normalization of X in $F(\theta)$, a tamely ramified cover by Lemma 5.3. Then $Y/\mathbb{Z}_p$ satisfies (2.4) by Lemma 3.1.

We claim $\mathrm{ind}(\alpha_{C_Y},\theta_{C_Y}) = \ell^{-1}\mathrm{ind}(\alpha_C,\theta_C)$. By Lemma 5.3 we have $\kappa(C_Y) = \kappa(C)(\phi_C)$, and in particular $\kappa(C_{i,Y}) = \kappa(C_i)(\phi_{C_i})$ is a field. Since π has multiplicity one at each irreducible component of C_Y by Lemma 3.1, we compute

$$\mathrm{ind}(\delta_{C_{i,Y}}) = |(\theta_{C_i})_{\kappa(C_i)(\phi_{C_i})}|\mathrm{ind}((\alpha_{C_{i,\kappa(C_i)(\theta_{C_i})}})_{\kappa(C_i)(\phi_{C_i})})$$

By construction, restriction of each nonzero δ_{C_i} to $\kappa(C_i)(\phi_{C_i})$ either lowers the order of θ_{C_i} by ℓ, if $\theta_{C_i} \neq 0$, or otherwise, since $\kappa(C_i)$ is a global field, lowers the index of $\alpha_{C_{i,\kappa(C_i)(\theta_{C_i})}}$ by ℓ (by the local-global splitting principle in class field theory). Therefore $\mathrm{ind}(\delta_{C_{i,Y}}) = \ell^{-1}\mathrm{ind}(\delta_{C_i})$, hence

$$\mathrm{ind}(\alpha_{C_Y},\theta_{C_Y}) = \ell^{-1}\mathrm{ind}(\alpha_C,\theta_C)$$

This proves the claim.

Since Y satisfies (2.4) we have $\mathrm{ind}(\lambda_Y(\alpha_{C_Y},\theta_{C_Y})) = \mathrm{ind}(\alpha_{C_Y},\theta_{C_Y})$ by the induction hypothesis, and $\lambda_Y(\alpha_{C_Y},\theta_{C_Y}) = \lambda(\alpha_C,\theta_C)_{K(Y)}$ by the commutative diagram of Lemma 4.12. Therefore $\mathrm{ind}(\lambda(\alpha_C,\theta_C)_{K(Y)}) = \ell^{-1}\mathrm{ind}(\alpha_C,\theta_C)$, hence

$$\mathrm{ind}(\lambda(\alpha_C,\theta_C)) \leq [K(Y):F]\mathrm{ind}(\lambda(\alpha_C,\theta_C)_{K(Y)}) = \ell\cdot\ell^{-1}\mathrm{ind}(\alpha_C,\theta_C) = \mathrm{ind}(\alpha_C,\theta_C)$$

On the other hand, $\lambda(\alpha_C,\theta_C)_{F_{C_i}} = \delta_{C_i}$ has index $\mathrm{ind}(\alpha_C,\theta_C)$ for some i by definition, hence $\mathrm{ind}(\lambda(\alpha_C,\theta_C)) \geq \mathrm{ind}(\alpha_C,\theta_C)$. Therefore we have equality, proving the theorem. □

Though we do not know how to lift all of $\mathrm{H}^2(F_C,\mu_n)$ to $\mathrm{H}^2(F,\mu_n)$, we now have the following.

Corollary 5.6. *Assume the setup of Theorem 5.5. Let* $\mathrm{G}^2_\pi \leq \mathrm{H}^2(F_C,\mu_n)$ *denote the image of* η_π *in* (4.13). *Then* $\mathrm{G}^2_\pi \simeq \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}},\mu_n) \oplus \Gamma$, *where* $\Gamma \leq \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)$ *is as in Lemma 4.6, and the map* λ *of Corollary 4.14 factors through* $\mathrm{H}^2(F_C,\mu_n)$ *to induce an injection*

$$\lambda : \mathrm{G}^2_\pi \longrightarrow \mathrm{H}^2(F,\mu_n)$$

that preserves index and splits the restriction $\mathrm{res}_{F|F_C}$.

Proof. We have $\ker(\eta_\pi) = \mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)$ by Lemma 4.6(b), hence in the notation of Lemma 4.6 we have an isomorphism $\mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}},\mu_n) \oplus \Gamma \simeq \mathrm{G}^2_\pi$. Since $\lambda(\ker(\eta_\pi)) = 0$ by the base case of the induction in Theorem 5.5, we have an induced map $\lambda : \mathrm{G}^2_\pi \longrightarrow \mathrm{H}^2(F,\mu_n)$; it preserves index by Theorem 5.5, and splits the restriction map by the commutative diagram in Theorem 4.9. □

6. Noncrossed Products and Indecomposable Division Algebras

We apply Theorem 5.5 to prove the existence of noncrossed product and indecomposable division algebras over the function field F of any p-adic curve. Noncrossed products over $K(t)$ for K a local field were first constructed in [8], and then constructed more systematically over the function field of a smooth relative $\mathbb{Z}_p$-curve in [10]. Indecomposable division algebras of unequal period and index were also constructed in [10], over the same types of fields. Modulo gluing, the method we use below is the same as the one used in [10, Theorem 4.3, Corollary 4.8]. We first present some background; for additional discussion, see e.g. [5], [2], or [38].

Let $\mathrm{Z}^2(\mathrm{G}_F, F_{\mathrm{sep}}^*)$ denote the group of continuous Galois 2-cocycles. For each f there exists a unique maximal open normal subgroup $U \lhd \mathrm{G}_F$ such that $f(sU, tU) = f(s,t)$ for all $s, t \in \mathrm{G}_F$. If L/F is the finite Galois extension with group $G = \mathrm{G}_F/U$, then f is inflated from $\mathrm{Z}^2(G, L^*)$, and f defines a central simple F-algebra A_f of degree $[L : F]$ via the the *crossed product construction*:

$$A_f = \sum_{s\in G} Lu_s : \ u_s u_t = f(s,t)u_{st}, \ \ u_s x = s(x)u_s, \ \ \forall x \in L$$

Here the u_s's are formal basis elements indexed by G. Let $\mathrm{CSA}(F)$ be the set of F-isomorphism classes of central simple F-algebras. The assignment $f \mapsto A_f$ defines a set map

$$\rho : \mathrm{Z}^2(\mathrm{G}_F, F_{\mathrm{sep}}^*) \longrightarrow \mathrm{CSA}(F)$$

It can be shown that L is a maximal commutative (étale) subalgebra of A_f, and we define a *crossed product* to be any central simple F-algebra with a Galois maximal commutative étale subalgebra. The *crossed product problem* over F is to determine whether every central simple F-algebra is a crossed product. In particular the problem is to determine whether every F-*division algebra* is a crossed product, or, in other words, whether there exist *noncrossed product* F-division algebras. It was long believed that all central simple algebras were crossed products, until Amitsur discovered noncrossed products in [1]. The set map ρ induces a surjective group homomorphism $\mathrm{Z}^2(\mathrm{G}_F, F_{\mathrm{sep}}^*) \to \mathrm{Br}(F)$ with kernel $\mathrm{B}^2(\mathrm{G}_F, F_{\mathrm{sep}}^*)$, giving the well-known isomorphism

$$\mathrm{H}^2(\mathrm{G}_F, F_{\mathrm{sep}}^*) \xrightarrow{\sim} \mathrm{Br}(F)$$

It follows by Wedderburn's theorem that for every F-division algebra D there exists a number r such that $\mathrm{M}_r(D)$ is a crossed product.

We say an F-division algebra D is *indecomposable* if it does not properly contain a nontrivial F-subalgebra that is also central over F, or equivalently if it is not an F-tensor product of two nontrivial F-division algebras. It is not hard to show that all division algebras of composite period are decomposable into their primary components, and that all division algebras of equal prime-power period and index are indecomposable. The first indecomposable division algebras of unequal prime-power period and index appeared in [3] and in [37]. These results showed that decomposability does not fully "account" for the phenomenon of unequal period and index in division algebras.

Theorem 6.1. *Let $F/\mathbb{Q}_p$ be a finitely generated field extension of transcendence degree one. Let $X/\mathbb{Z}_p$ be a regular relative curve with function field F, let C_1 be a reduced irreducible component of the closed fiber, let $\ell \neq p$ be a prime, and let r*

and s be numbers that are maximal such that $\mu_{\ell^r} \subset \kappa(C_1)$ and $\mu_{\ell^s} \subset \kappa(C_1)(\mu_{\ell^{r+1}})$. Then there exist noncrossed product F-division algebras of period and index as low as ℓ^{s+1} if $r = 0$, and ℓ^{2r+1} if $r \neq 0$.

Proof. We may assume (without changing r and s) that C has regular irreducible components, at most two of which meet at any closed point of X. The idea is to use the (known) existence of such algebras over F_{C_1} to produce a class in G^2_π, and then apply Corollary 5.6 to prove existence over F.

By [10, Theorem 4.7], if F admits a smooth model X then there exist noncrossed product division algebras over F_C of period and index as low as ℓ^{s+1} if $r = 0$, and ℓ^{2r+1} if $r > 0$. The resulting Brauer class has the form $\alpha_C + (\pi)\cdot\theta_C \in \mathrm{H}^2(F_C, \mu_n)$, where $\alpha_C \in \mathrm{H}^2(\kappa(C), \mu_n)$ and $\theta_C \in \mathrm{H}^1(\kappa(C), \mathbb{Z}/n)$. A look at the construction, which proceeds exactly as in [6, Theorem 1], shows we may pre-assign values at any finite set of points of C. Thus we may produce a noncrossed product F_{C_1}-division algebra $D_{F_{C_1}}$ with class $\delta_{C_1} = \alpha_{C_1} + (\pi)\cdot\theta_{C_1}$ of the desired period and index, such that $\alpha_{C_1,z} = 0$ and $\theta_{C_1,z} = 0$ at all $z \in \mathcal{S} \cap C_1$. Set $\alpha_{C_i} = 0$ and $\theta_{C_i} = 0$ for $i > 1$. Then $\alpha_C = (\alpha_{C_i})_i \in \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}, \mu_n)$ and $\theta_C = (\theta_{C_i})_i \in \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ by Lemma 4.2, and we have an element

$$\delta_C = \alpha_C + (\pi)\cdot\theta_C \ \in \mathrm{G}^2_\pi \leq \mathrm{H}^2(F_C, \mu_n)$$

whose restriction to F_{C_1} is δ_{C_1}. Then δ_C lifts to $\delta = \lambda(\delta_C) \in \mathrm{H}^2(F, \mu_n)$, and $\mathrm{ind}(\delta) = \mathrm{ind}(\delta_C) = \mathrm{ind}(\delta_{C_1})$ by Corollary 5.6 and the definition of index over $\kappa(C)$. Let D be the F-division algebra associated to δ. Since $\mathrm{res}_{F|F_{C_1}}(\delta) = \delta_{C_1}$ (also by Corollary 5.6) and $\mathrm{ind}(\delta) = \mathrm{ind}(\delta_{C_1})$, $D \otimes_F F_{C_1}$ is the (noncrossed product) F_{C_1}-division algebra $D_{F_{C_1}}$ associated to δ_{C_1}. If L/F is a Galois maximal subfield of D then $L \otimes_F F_{C_1}/F_{C_1}$ is a Galois maximal subfield of $D_{F_{C_1}}$ by degree count. Since $D_{F_{C_1}}$ is a noncrossed product this would be a contradiction, and we conclude D has no Galois maximal subfields. Therefore D is a noncrossed product F-division algebra. □

Theorem 6.2. *Let $F/\mathbb{Q}_p$ be a finitely generated field extension of transcendence degree one, and let $\ell \neq p$ be a prime. Then there exist indecomposable F-division algebras of (period,index)= (ℓ^a, ℓ^b), for any numbers a and b satisfying $1 \leq a \leq b \leq 2a - 1$.*

Proof. Let X, C, C_i, and $\mathcal{S}$ be as in (2.4). The construction over F_{C_1} proceeds exactly as in [10, Proposition 4.2] and [7], and we obtain an indecomposable F_{C_1}-division algebra $D_{F_{C_1}}$ of (period,index)= (ℓ^a, ℓ^b), for any numbers a and b satisfying $1 \leq a \leq b \leq 2a - 1$. The division algebra $D_{F_{C_1}}$ is associated to a class $\delta_{C_1} = \alpha_{C_1} + (\pi)\cdot\theta_{C_1} \in \mathrm{H}^2(F_{C_1}, \mu_n)$. The construction of [10] allows us to assume the components α_{C_1} and θ_{C_1} are zero at the singular points $\mathcal{S}\cap C_1$, so that by Lemma 4.2 we obtain a class $\delta_C = \alpha_C + (\pi)\cdot\theta_C$ in $\mathrm{G}^2_\pi \leq \mathrm{H}^2(F_C, \mu_n)$ whose first component is $\delta_{C_1} = \alpha_{C_1} + (\pi)\cdot\theta_{C_1}$, and whose other components are zero. This class lifts to a class $\delta = \lambda(\delta_C)$, and $\mathrm{ind}(\delta) = \mathrm{ind}(\delta_C)$ and $\delta_C = \mathrm{res}_{F|F_C}(\delta)$ by Corollary 5.6, hence $\mathrm{ind}(\delta) = \mathrm{ind}(\delta_{C_1})$ and $\delta_{C_1} = \mathrm{res}_{F|F_{C_1}}(\delta)$ by definition of index and Brauer class over F_C. Let D be the F-division algebra associated to δ. Since $\mathrm{ind}(\delta) = \mathrm{ind}(\delta_{C_1})$ $D \otimes_F F_{C_1}$ is isomorphic to the F_{C_1}-division algebra $D_{F_{C_1}}$ associated to δ_{C_1}. If D is decomposable then $D \simeq D_1 \otimes_F D_2$ for nontrivial F-division algebras D_1 and D_2, hence $D_{F_{C_1}} = (D_1)_{F_{C_1}} \otimes_{F_{C_1}} (D_2)_{F_{C_1}}$ is a decomposition of $D_{F_{C_1}}$. Since $D_{F_{C_1}}$ is a division algebra, both factors $(D_i)_{F_{C_1}}$ are nontrivial, hence $D_{F_{C_1}}$ is decomposable, a contradiction. We conclude D is indecomposable. □

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