Cal Poly

Algebra Qualifying Exam

September 18, 2021

Directions.

- **Show all work!** Write everything down. Insufficient justification can mean no credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators and phones

****

1. Let n be a number between 0 and 10. Compute n^{111} (mod 11), expressing your answer as a number between 0 and 10. Give as detailed a proof as you can, justifying every step, no matter how trivial you think it is.

2. Let G be a group of order $2n$ for some positive integer $n > 1$.

(a) Prove there exists a subgroup K of G of order 2.

(b) Suppose K in (a) is a *normal* subgroup. Prove that K is contained in the center $Z(G)$. (Recall $Z(G) = \{a \in G : ab = ba \ \forall b \in G\}$)

3. Consider the additive group of integers $\mathbb{Z}$.

(a) Prove that every subgroup of $\mathbb{Z}$ is a cyclic group.

(b) Prove that every homomorphic image of $\mathbb{Z}$ is a cyclic group.

(c) Now consider the *ring* $\mathbb{Z}$. Exhibit a prime ideal of $\mathbb{Z}$ that is not maximal.

4. Let $i \in \mathbb{C}$ be the usual root of unity, with $i^2 = -1$, and let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers.

(a) Prove that there exists a (nonzero) ring homomorphism $\mathbb{Z}[i] \longrightarrow \mathbb{Z}_5$.

(b) Compute the kernel of your homomorphism explicitly, and state the conclusion given by the First Isomorphism Theorem. [*Hint*: The kernel requires two generators.]

5. Let a and b be real numbers and let $A \in \mathbb{R}^{3\times3}$ with each diagonal entry equal to a and each off-diagonal entry equal to b.

(a) Determine all eigenvalues and representative eigenvectors of A together with their algebraic multiplicities. [*Hint*: $A = (a - b)I + bJ$ where J is the 3×3 matrix each of whose entries equals 1.]

(b) Is A diagonalizable? Justify your answer.

(c) Determine the minimal polynomial of A.