# Real Analysis Qualifying Exam, September 19, 2021

**Instructions**: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Good luck!

1. Let $f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{18}+x^2}$. Show $f$ is a continuous function on $\mathbb{R}$.

2. Let $a \geq 0$. Define the sequence $(x_n)$ by $x_0 = 0$ and $x_{n+1} = a + x_n^2$ for all $n \geq 0$. Find, with proof, a necessary and sufficient condition on $a$ for the sequence to converge, and for those values of $a$ which make the sequence converge, find the limit.

3. Let $f : [0,1] \to [0,1]$ be continuous. Show there is a number $c \in [0,1]$ with $f(c) = c$.

4. Let $f$ and $g$ be real valued functions continuous on $[a,b]$ and differentiable on $(a,b)$, with $f(a) = f(b) = 0$. Show there is a point $c$ in $(a,b)$ with

$$f'(c) + g'(c)f(c) = 0.$$

5. (a) State a definition for a real valued function $f : [a,b] \to \mathbb{R}$ to be Riemann integrable.

(b) Let $f$ and $g$ be Riemann integrable on $[a,b]$ and define $h(x) = \max\{f(x), g(x)\}$ for each $x$ in $[a,b]$. Show, using your definition, that $h$ is also Riemann integrable on $[a,b]$.

**Note**: If you choose to work with a definition of Riemann integrability different from that stated in part (a), please provide the alternate definition.