Real Analysis Qualifying Exam, June 4, 2023

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Good luck!

1. For each $n = 1, 2, 3, \ldots$ $f_n(x) = \frac{nx}{e^{nx}}$ is a continuous function on $[0, 2]$. Find the pointwise limit function $f(x) = \lim_{n\to\infty} f_n(x)$ and show that $f_n(x)$ does not converge uniformly to $f(x)$.

2. Let f be Riemann integrable on $[0, 1]$ with $f(x) > 0$ for all x in $[0, 1]$. Prove that $\int_0^1 f(x)dx > 0$.

3. Let f be continuous on $[0, 1]$ with $f(x) > 0$ for all x. Let $S = \sup\{f(x) : x \in [0, 1]\}$. Show that for every $\varepsilon > 0$ there is some open interval I on which $f(x) > S - \varepsilon$.

4. Prove that if f is a function which is differentiable on all of $\mathbb{R}$, and $f'(x) > 0$ for all x, then f is injective.

5. Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$. Prove that (x_n) converges.