Real Analysis Qualifying Exam, January 22, 2023

**Instructions**: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Good luck!

1. Show that if $f_n(x)$ is a uniformly continuous function on $[0,1]$ for each $n = 1, 2, 3, \ldots$ and $\{f_n(x)\} \to f(x)$ uniformly on $[0,1]$, then $f(x)$ is also uniformly continuous on $[0,1]$.

2. Show that the sequence $\{x_n\}$ is Cauchy, where

$$x_n = \int_1^n \frac{\cos t}{t^2} dt.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with

$$\int_0^1 f(xt)dt = 0 \text{ for all } x \in \mathbb{R}.$$

Show $f(x) \equiv 0$.

4. Suppose $f$ and $g$ are continuous on $[a,b]$ and $f'$ and $g'$ are continuous on $(a,b)$ with $f(a) = g(a)$ and $f(b) = g(b)$. Prove there is a number $c$ in $(a,b)$ such that the line tangent to the graph of $f$ at the point $(c, f(c))$ is parallel to the line tangent to the graph of $g$ at $(c, g(c))$.

5. Prove that $f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$ is continuous on $\mathbb{R}$.