Real Analysis Qualifying Exam, September 17, 2023

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Good luck!

1. Suppose that $f(x)$ is continuous and unbounded on $[a, b)$. Prove that $\lim_{x \to b^-} f(x)$ does not exist.

2. Prove that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 + x^4}{n^4 + x^2}$$

converges to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

3. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence with $a_n \geq 0$ for all n. Let $x_1 = 1$ and let

$$x_{n+1} = \frac{1}{2}\left(x_n + \sqrt{x_n^2 + a_n}\right)$$

for $n \geq 1$.

(a) Prove that $x_{n+1} \leq x_n + \frac{a_n}{4}$ for all $n \geq 1$.
(b) Deduce that (x_n) converges whenever $\sum_{n=1}^{\infty} a_n$ converges.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and twice differentiable on (a, b). Assume that the line segment from $A = (a, f(a))$ to $B = (b, f(b))$ intersects the graph of f in a third point different from A and B. Show that $f''(c) = 0$ for some $c \in (a, b)$.

5. (a) State a definition for a real valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

(b) Use this definition (or another one which you state clearly) to prove that the function f defined on $\left[0, \frac{\pi}{2}\right]$ by

$$f(x) = \begin{cases} \cos^2 x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is not Riemann integrable.