Real Analysis Qualifying Exam, September 18, 2022

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Good luck!

1. Let $f_n(x) = \frac{x}{(x+\cos(x/n))^n}$ for each $n = 1, 2, 3, \ldots$. Prove that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is continuous on $[1, 2]$.

2. Let $\{x_n\}$ be a sequence of real numbers with $x_1 > 0$ and $x_{n+1} = \frac{1}{x_n + \frac{1}{x_n}}$ for all $n \geq 1$. Prove $\{x_n\}$ converges and find its limit.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $f'(x) \neq 1$ for all $x \in \mathbb{R}$. Show f has at most one fixed point. Note: A fixed point of a function f is a number ξ with $f(\xi) = \xi$.

4. Consider all positive term series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ with $\frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n}$ for all $n = 1, 2, 3, \ldots$, that is, $\frac{x_2}{x_1} \leq \frac{y_2}{y_1}, \frac{x_3}{x_2} \leq \frac{y_3}{y_2}, \ldots, \frac{x_n}{x_{n-1}} \leq \frac{y_n}{y_{n-1}}, \frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n}, \ldots$ Show that if $\sum_{n=1}^{\infty} x_n$ is divergent, then so is $\sum_{n=1}^{\infty} y_n$.

5. (a) State a definition for a real valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

(b) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Prove that $|f(x)|$ is also Riemann integrable and

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\, dx.$$