Cal Poly | **Algebra Qualifying Exam** | **January 2024**

Directions

- **Show all work!** Write everything down. Insufficient justification can mean no credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators and phones.

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Groups

G1. (5 points) Let G be a group, $H \leq G$ a subgroup that is not normal. Prove there exist cosets Ha and Hb such that $HaHb \neq Hab$.

G2. (5 points) Determine with proof the automorphism group $\mathsf{Aut}(V)$ of the Klein 4-group $V = \{e, a, b, ab\}$. To what familiar group is it isomorphic?

Rings

R1. (5 points) Suppose R is a finite ring with no nontrivial zero-divisors. Prove that R contains an element 1 satisfying $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

R2. (5 points) Let $k \subset K$ be fields, and let $k[X]$ be the polynomial ring in one variable with coefficients in k. The *evaluation* at $z \in K$ is a ring homomorphism $\varepsilon : k[X] \longrightarrow K$ defined by $\varepsilon(f(X)) = f(z)$. Prove that if ε is not injective, then $\varepsilon(k[X])$ is a field.

Vector Spaces

VS 1. (5 points) Let V be a vector space with basis $\mathbf{v}_0, \ldots, \mathbf{v}_n$ and let $a_0, \ldots, a_n$ be scalars. Define a linear transformation $T : V \longrightarrow V$ by the rules $T(\mathbf{v}_i) = \mathbf{v}_{i+1}$ if $i < n$, and $T(\mathbf{v}_n) = a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. You don't have to prove this defines a linear transformation. Determine the matrix of T with respect to the basis $\mathbf{v}_0, \ldots, \mathbf{v}_n$, and determine the characteristic polynomial of T.