Cal Poly | **Algebra Qualifying Exam** | **January, 2023**

- **Show all work!** Write everything down. Insufficient justification can mean no credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators and phones.

****

Groups

1. (5 points) Let G be a finite group and $n > 1$ an integer such that $(ab)^n = a^n b^n$ for all $a, b \in G$. Let

$$G_n = \{c \in G : c^n = e\} \qquad \text{and} \qquad G^n = \{c^n : c \in G\}$$

You may take for granted that these are subgroups. Prove that both G_n and G^n are normal in G, and $|G^n| = [G : G_n]$.

2. (5 points) Show that every finite group with more than two elements has a nontrivial automorphism.

Rings

3. (5 points) Let R be a commutative ring with identity. Suppose that for every $a \in R$ there is an integer $n \geq 2$ such that $a^n = a$. Show that every prime ideal of R is maximal.

Linear Spaces

4. (5 points) Let $\mathrm{M}_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real entries. We say that $A, B \in \mathrm{M}_n(\mathbb{R})$ commute if $AB = BA$.

(a) Fix $A \in \mathrm{M}_n(\mathbb{R})$. Prove that the set of all matrices in $\mathrm{M}_n(\mathbb{R})$ that commute with A is a subspace of $\mathrm{M}_n(\mathbb{R})$.

(b) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$ and let $W \subseteq \mathrm{M}_2(\mathbb{R})$ be the subspace of all matrices of $\mathrm{M}_2(\mathbb{R})$ that commute with A. Find a basis of W.

5. (5 points) Let $S : V \to V$ and $T : V \to V$ be linear transformations that commute, i.e. $S \circ T = T \circ S$. Let $v \in V$ be an eigenvector of S such that $T(v) \neq 0$. Prove that $T(v)$ is also an eigenvector of S.