Cal Poly # Algebra Qualifying Exam **January, 2023**

- **Show all work!** Write everything down. Insufficient justification can mean loss of credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators and phones.

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Groups

1. (5 points) Let G denote the set of invertible 2×2 matrices with values in a field. Prove G is a group by defining a group law, identity element, and verifying the axioms. Credit is based on completeness.

2. (5 points) Let G be a finite group. Prove *from the definitions* that there exists a number N such that $a^N = e$ for all $a \in G$.

Rings

3. (5 points) Suppose R is a PID (principal ideal domain). Prove that an ideal $I \subset R$ is maximal if and only if $I = (p)$ for a prime $p \in R$. (By definition, an element p is *prime* if whenever $p \mid ab$ then $p \mid a$ or $p \mid b$. If you use the fact that prime implies irreducible, you have to prove it.)

4. (5 points) Let $\mathscr{C}([0,1])$ be the (commutative) ring of continuous, real-valued functions on the unit interval, and let $M = \{f \in \mathscr{C}([0,1]) \ : \ f(\frac{1}{2}) = 0\}$. Prove that M is a maximal ideal.

Vector Spaces

5. (5 points) Suppose V is a vector space, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are in V. Prove that either $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, or there exists a number $k \leq n$ such that $\mathbf{v}_k$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.