Cal Poly | **Algebra Qualifying Exam** | **September, 2023**

Directions

- **Show all work!** Write everything down. Insufficient justification can mean no credit.
- Start each problem on a new page.
- No assistance of any kind is allowed on this exam. This includes calculators and phones.

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Groups

G1. (5 points) Let G be a group, and let $\mathrm{Aut}(G)$ denote the group of automorphisms of G. There is a homomorphism $\gamma : G \longrightarrow \mathrm{Aut}(G)$ that takes $s \in G$ to the automorphism γ_s defined by $\gamma_s(t) = sts^{-1}$.

(a) Prove rigorously, possibly with induction, that if $\gamma_s(t) = t^b$, then $\gamma_{s^n}(t) = t^{b^n}$.

(b) Suppose $s \in G$ has order 5, and $sts^{-1} = t^2$. Find the order of t. Justify your answer.

G2. (5 points) Suppose G is a nonempty finite set that has an associative pairing $G \times G \longrightarrow G$, written $(x, y) \mapsto x \cdot y$. Suppose this pairing satisfies left and right cancellation: $x \cdot y = x \cdot y'$ implies $y = y'$, and $x \cdot y = x' \cdot y$ implies $x = x'$. Prove there exists an element e of G such that for all $x \in G$, $e \cdot x = x \cdot e = x$. Justify your reasoning as carefully as possible.

Rings

R1. (5 points) Let $R_1, \ldots, R_k$ be commutative rings, and set $R = R_1 \times \cdots \times R_k$.

(a) Let $I_j \subset R_j$ be ideals, and put $I = I_1 \times \cdots \times I_k$. Use the first isomorphism theorem to prove that $R/I \simeq R_1/I_1 \times \cdots \times R_k/I_k$.

(b) Prove the prime ideals of R have the form $R_1 \times \cdots \times R_{j-1} \times P_j \times R_{j+1} \times \cdots \times R_k$ where $P_j \subset R_j$ is a prime ideal for $1 \leq j \leq k$. (Omit the proof that this is an ideal.)

R2. (5 points) Let i be the imaginary number, let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, a principal ideal domain, and let $\mathbb{Z}_2$ be the finite ring of integers $\pmod 2$.

(a) Define a ring homomorphism from $\mathbb{Z}[i]$ to $\mathbb{Z}_2$. You must prove it is a ring homomorphism.

(b) Find, with proof, a generator for the kernel of your ring homomorphism.

Vector Spaces

VS 1. (5 points) Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be corresponding eigenvectors. Prove $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.