2018 Summer Reseach Project
Measure-geomtric Laplacians – Fractal Derivatives

Supervisors

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Project Description

The aim of this project is examine new ways of taking derivatives, in particular we will investigate opertors which have become to be known as measure-geometric Laplacians.

Motivated by the fundamental theorem of calculus, and based on the works of Feller, Kac and Kreĭn, given an atomless Borel probability measure η supported on a subset of the unit interval $[0, 1]$, Freiberg and Zähle introduced a measure-geometric approach to define a first order differential operator ∇_η and a second order differential operator $\Delta_{\eta,\eta} := \nabla_\eta \circ \nabla_\eta$, with respect to η. In the case that η is the Lebesgue measure, it was shown that ∇_η coincides with the (weak) derivative we know from Calculus. Moreover, a harmonic calculus for $\Delta_{\eta,\eta}$ was developed; focusing on the case when η is a self-similar measure supported on a Cantor set. In 2016, Kesseböhmer, Samuel and Weyer gave the exact eigenvalues and eigenfunctions of $\Delta_{\eta,\eta}$ and it was shown that the eigenvalues do not depend on the given measure. Interestingly, the operator $\Delta_{\eta,\eta}$, in the case that η is continuous, has an intimate relationship to Liouville Brownian motion.

Last year, Kesseböhmer, Samuel and Weyer showed that the framework of Freiberg and Zähle can be extended to include purely atomic measures η. Unlike in the case when one has a absolutely continuous measure, it was proven that the operators ∇_η and $\Delta_{\eta,\eta}$ are no longer symmetric. To circumvent this problem, the η-Laplacian was defined by $\Delta_\eta = -\nabla_\eta^* \circ \nabla_\eta$, where ∇_η^* denotes the adjoint of ∇_η. Here a matrix representations for these operators was given and shown to coincide with the normalised graph Laplacian of a cycle graph; and that the eigenvalues of Δ_η depend only on the weights of the atoms and are independent of the positions of the atoms.

In the first part of the project we will focus on the case when η is purely atomic, with a finite number of atoms with the hope of extending the results of Kesseböhmer, Samuel and Weyer. The second part will be to examine the case when η is an atomic measure supported on the boundary of a fractal string.

Reference

- M. Kesseböhmer, T. Samuel and H. Weyer. Measure-geometric Laplacians on the real line: Pre-print: arXiv:1802.04858.
- M. Kesseböhmer, T. Samuel and H. Weyer. Measure-geometric Laplacians for discrete distributions: Accepted for publication in *Contemp. Math.* 8 pages (2017).
- M. Kessebhmer T. Samuel and H. Weyer. A note on measure-geometric Laplacians: *Monatsh. Math.* **181**(3), 643-655 (2016).

Requirements

Required:	A strong background in real analysis.
Desirable:	A basic understanding of measure theory and topology. Knowledge of Matlab and/or Mathematica.