Dear Students,

Thanks for your interest in reading this. My name is Caixing Gu. My research is in the areas of operator theory, linear algebra and complex analysis. I studied general bounded linear operators on Hilbert spaces and Banach spaces. These operators can often be represented as infinite matrices. I also studied more concrete operators such as Toeplitz, Hankel operators, composition operators on various function spaces. The essential background needed for summer projects is linear algebra (such as Math 306). Abstract algebra (such as Math 481) and complex analysis (such as Math 408) are also useful but not essential. The tentative schedule for the project is one to two weeks from June 18, 2018 to June 29, plus six to seven weeks from July 30 to September 14. The total duration is eight weeks. I will describe one potential project in some details, please feel free to talk to me about other possible projects or to suggest your own projects.

Reducing subspaces of tensor product of matrices and group representation

In operator theory, in general we study linear operators on infinite dimensional Hilbert spaces and more general Banach spaces. In this project we will restrict ourself to finite dimensional spaces.

An example of a finite dimensional Hilbert space is our familiar n-dimensional Euclidean space $\mathbb{C}^n$ with the dot product where $\mathbb{C}$ denotes the set of complex numbers. For example in $\mathbb{C}^3$, if

$$u = \begin{bmatrix} 1 \\ -3-i \\ 5 \end{bmatrix}, v = \begin{bmatrix} 4-i \\ 2+i \\ 0 \end{bmatrix},$$

then the dot product (or inner product) of u and v is

$$\langle u, v\rangle = u \bullet v = 1 \times \overline{(4-i)} + (-3-i) \times \overline{(2+i)} + 5 \times \overline{0} = 4 + i - 7 + i + 0 = -3 + 2i$$

where $\overline{(4-i)} = 4 + i$ is the conjugate of the complex number $4 - i$.

A linear operator A on $\mathbb{C}^n$ under the standard basis is simply a $n \times n$ matrix, still denoted by A. The adjoint of A, denoted by A^*, is defined to be the linear operator satisfying

$$\langle Au, v\rangle = \langle u, A^*v\rangle \text{ for all } u, v \in \mathbb{C}^n.$$

The matrix of A^* is the transpose and conjugate of the matrix A.

A subspace H_0 of $\mathbb{C}^n$ is invariant for A if A maps H_0 back into H_0. In this case,

$$A \text{ (under a different basis of } \mathbb{C}^n) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} : H_0 \oplus H_0^\perp \to H_0 \oplus H_0^\perp$$

where $H_0^\perp = \mathbb{C}^n \ominus H_0$ is the orthogonal complement of H_0 in $\mathbb{C}^n$. Equivalently for some invertible matrix S,

$$A \text{ (under the standard basis of } \mathbb{C}^n) = S^{-1} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} S.$$

A subspace H_0 of $\mathbb{C}^n$ is reducing for A if H_0 is invariant for both A and A^*. In this case, for some unitary matrix S (S is unitary if $U^*U = I$)

$$A \text{ (under the standard basis of } \mathbb{C}^n) = S^* \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix} S.$$

As you can see, to find the invariant subspaces of A and reducing subspaces of A is to decompose A into a block upper triangular or diagonal matrix. This is a fundamental problem in operator theory as in many scientific research where one tries to break a complicated object into several simple parts.

In this project, we will try to find the invariant subspaces and reducing subspaces of tensor products of matrices. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $n \times n$ matrices. The tensor product (or Kronecker product) of A and B, denote by $A \otimes B$, is the $n^2 \times n^2$ block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

In general $A \otimes B \neq B \otimes A$.

For any two matrices A and B, we can check the following two subspaces of $\mathbb{C}^n \otimes \mathbb{C}^n$ are reducing for $A \otimes B + B \otimes A$.

$$\begin{aligned} H_s &= \text{Span}\{v_1 \otimes v_2 + v_2 \otimes v_1 : v_1, v_2 \in \mathbb{C}^n\}, \\ H_{as} &= \text{Span}\{v_1 \otimes v_2 - v_2 \otimes v_1 : v_1, v_2 \in \mathbb{C}^n\}, \end{aligned}$$

where H_s is the subspace of symmetric tensors and H_{as} is the subspace of anti-symmetric tensors. Furthermore

$$\mathbb{C}^n \otimes \mathbb{C}^n = H_s \oplus H_{as}.$$

The reason H_s and H_{as} are reducing for $A \otimes B + B \otimes A$ is that $A \otimes B + B \otimes A$ has the permutation symmetry. The permutation of $A \otimes B$ is $B \otimes A$, so the permutation of $A \otimes B + B \otimes A$ is itself.

Problem 1 *For which A and B, H_s and H_{as} are the only reducing subspace for $A \otimes B + B \otimes A$?*

We do not have to restrict to the tensor product of two matrices. For example, we can try to find the reducing subspaces of operator T where

$$T := A \otimes B \otimes C + B \otimes C \otimes A + C \otimes A \otimes B.$$

We notice that T has a permutation symmetry for some permutation in the permutation group of three elements. It turns out the group representation theory of permutation group will provide us a first step for finding reducing subspaces of T. The representation theory of permutation group and more generally the representation theory of finite groups and Lie groups have been studied intensively in the past because their importance in mathematics and physics (nowadays in quantum information and quantum computing).