Algebra Qualifying Exam

March 12, 2016

Do all five problems.

1. Without using Cauchy's Theorem or the Sylow theorems, prove that every group of order 21 contains an element of order 3.

2. Suppose G is a group that contains normal subgroups $H, K \trianglelefteq G$ with $H \cap K = \{e\}$ and $HK = G$. Prove that $G \cong H \times K$.

3. Let R be a commutative ring.

   (a) Prove that the set N of all nilpotent elements of R is an ideal.

   (b) Prove that R/N is a ring with no nonzero nilpotent elements.

   (c) Show that N is contained in every prime ideal of R.

4. Let $z \in \mathbb{C}$ be a complex number and let $\epsilon_z : \mathbb{R}[x] \to \mathbb{C}$ be the evaluation homomorphism given by $\epsilon_z(p(x)) = p(z)$ for each $p(x) \in \mathbb{R}[x]$.

   (a) Show that $\ker(\epsilon_z)$ is a prime ideal.

   (b) Compute $\ker(\epsilon_{1+i})$, $\operatorname{im}(\epsilon_{1+i})$ and then state the conclusion of the First Isomorphism Theorem applied to the homomorphism ϵ_{1+i}.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that expands radially by a factor of 3 around the line parameterized by $L(t) = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} t$, leaving the line itself fixed (viewed as a subspace).

   (a) Find an eigenbasis for T and provide the matrix representation of T with respect to that basis.

   (b) Provide the matrix representation of T with respect to the standard basis.