Real Analysis Qualifying Exam, January 12, 2020

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, periodic function. Prove that the set $f(\mathbb{R})$ is compact. (Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* if there exists a nonzero constant P such that $f(x) = f(x+P)$ for all $x \in \mathbb{R}$.)

2. Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ ... \ + \frac{1}{2n}.$$

(a) Prove that $\{a_n\}$ is increasing.

(b) Prove that $\{a_n\}$ converges.

3. Let $f : [-1,1] \to \mathbb{R}$ be continuous. Suppose that f is differentiable on $(-1,0) \cup (0,1)$ and that

$$\lim_{x \to 0} f'(x) = \alpha$$

for some $\alpha \in \mathbb{R}$. Show that $f'(0)$ exists and that $f'(0) = \alpha$.

4. Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(x^n)}{n^2 x^n}.$$

(a) Prove that f is continuous on $[1, \infty)$.

(b) Prove that, in fact, f is continuous on $(0, \infty)$.

5. Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) x^n \ dx = 0.$$