Real Analysis Qualifying Exam, June 7, 2020

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Let (f_n) be a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose f_n is bounded for each $n \in \mathbb{N}$.

(a) Prove that if $f_n \to f$ uniformly on $\mathbb{R}$, then f is bounded.

(a) If each f_n is continuous and $f_n \to f$ pointwise on $\mathbb{R}$, does f have to be bounded? Give a proof or a counter example.

2. Show that the function

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is differentiable only at $x = 0$.

3. Consider the sequence of functions

$$f_k(x) = \frac{x^k \left(\sin(kx^2 + 1) + \cos(\pi - 5x)\right)}{k!}.$$

Prove that the series $\sum_{k=0}^{\infty} f_k$ converges uniformly on any interval of the form $[-M, M]$ in $\mathbb{R}$.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is *Lipschitz* on a set $A \subseteq \mathbb{R}$ if there exists a constant $M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$.

(a) Assume that f is a differentiable function on $\mathbb{R}$ and that f' is continuous on $[a, b]$. Prove that f is Lipschitz on $[a, b]$.

(b) Prove that a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

5. (a) State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$.

(b) Define $f : [0, 4] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \in [0, 1) \\ 2, & x \in [1, 2) \\ 3, & x \in [2, 3) \\ 4, & x \in [3, 4] \end{cases}.$$

Use the definition of the Riemann integral to prove that f is Riemann integrable on $[0, 4]$.

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.