Real Analysis Qualifying Exam, June 10, 2018

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges by showing that the sequence of partial sums is Cauchy.

2. (a) Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. Prove that if each f_n is continuous at $c \in A$, then f is continuous at c.

(b) Give an example to show that the result above is false if we only assume that (f_n) converges pointwise to f on A.

3. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{x}{k(1+kx^2)}.$$

(a) Fix $\epsilon > 0$. Prove that f is continuous for $|x| \geq \epsilon$.

(b) Prove that, in fact, f is continuous on $\mathbb{R}$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $|f'(x)| < 1$ for all $x \in \mathbb{R}$.

(a) Prove that f has at most one fixed point.

(b) Show that the following function satisfies $|f'(x)| < 1$ for all $x \in \mathbb{R}$ but has no fixed points:

$$f(x) = \ln(1 + e^x)$$

5. (a) State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$.

(b) Use your definition from (a) to prove that if $f : [a, b] \to \mathbb{R}$ is continuous and

$$\int_a^b |f(x)| \, dx = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.