Real Analysis Qualifying Exam, June 8, 2019

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Show that the set $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \cup \left\{1 + \frac{1}{n} : n \in \mathbb{Z}^+\right\}$ is compact using the open-cover definition of compactness.

2. Show that if (a_n) is a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} a_n$ diverges, then

$$\lim_{n\to\infty} \frac{a_1 + a_3 + a_5 + \cdots + a_{2n-1}}{a_2 + a_4 + a_6 + \cdots + a_{2n}} = 1.$$

3. Show that $\sqrt{1-x} \leq 1 - \frac{1}{2}x - \frac{1}{8}x^2$ for all x in $[0, 1]$

4. Show that the following series converges uniformly on (r, ∞) for any real number $r > 1$.

$$\sum_{n=1}^{\infty} \frac{n \ln(1 + nx)}{x^n}$$

5. (a) State a definition for a real valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

(b) Let $f : [0, 1] \to \mathbb{R}$ be Thomae's function, defined by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ for relatively prime natural numbers } m \text{ and } n \\ 0 & \text{otherwise.} \end{cases}.$$

Show f is Riemann integrable.

Note: If you choose to work with a definition of Riemann integrability different from that stated in part (a), please provide the alternate definition.