Real Analysis Qualifying Exam, June 10, 2017

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Prove that every uncountable subset of $[0, \infty)$ contains a sequence $\{a_n\}$ of distinct points such that $\sum a_n$ diverges.

2. Let $f : [0, \infty) \to \mathbb{R}$ be a bounded, monotone increasing, continuous function. Prove that f is uniformly continuous on $[0, \infty)$.

3. Suppose f and g are differentiable functions on R such that $f'(x)g(x) \neq f(x)g'(x)$ for any $x \in \mathbb{R}$. Prove that between any two zeros of f (if any), there must lie a zero of g.

4. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} |x|, & -1 \leq x \leq 1, \\ g(x - 2n), & 2n - 1 \leq x \leq 2n + 1, \end{cases}$$

so that g is periodic with period 2. Prove that the following function is uniformly continuous on $\mathbb{R}$.

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g(4^n x).$$

5. (a) State the definition for a real valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a, b]$.

(b)Use the definition of the Riemann integral to prove that $f(x) = \frac{1}{1+x}$ is Riemann integrable on $[0, b]$, for any $b > 0$.

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.