Real Analysis Qualifying Exam, June 4, 2016

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. (a) Argue from the definition of Cauchy sequence that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is $\{a_n b_n\}$.

(b) Give an example of a sequence $\{a_n\}$ with $\lim |a_{n+1} - a_n| = 0$ but which is *not* Cauchy.

2. Let f be a function that is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Show that if $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in (0, 1)$, then $f(x) = 0$ for all $x \in [0, 1]$.

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions that converges uniformly on $\mathbb{R}$ to a function f. Let $\{x_n\}$ be a sequence of real numbers that converges to $x_o \in \mathbb{R}$. Prove that $\{f_n(x_n)\} \to f(x_o)$.

4. Let $P = \{2, 3, 5, 7, 11, 13, \ \dots \}$ be the set of prime numbers.

(a) Find the radius of convergence R of the power series

$$f(x) = \sum_{p \in P} x^p = x^2 + x^3 + x^5 + x^7 + \ \dots \ .$$

(b) Show that $0 \leq f(x) \leq \frac{x^2}{1-x}$ for $0 \leq x < R$.

5. (a) State the definition for a real valued function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on the interval $[a, b]$.

(b) Let $f : [a, b] \to \mathbb{R}$ be increasing on the inverval $[a, b]$. Use the definition to prove that f is Riemann integrable on $[a, b]$.

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.