Real Analysis Qualifying Exam, September 13, 2020

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Let $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + s_n}$ for $n = 1, 2, 3, \ \ldots$.

(a) Show that $s_n \leq 2$ for all n.

(b) Show that $\{s_n\}$ converges and then compute the limit of the sequence.

2. Define $f : (-1, 0) \cup (0, 1) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 4, & x \in (-1, 0) \\ 5, & x \in (0, 1) \end{cases}.$$

(a) Show that f is continuous on $(-1, 0) \cup (0, 1)$.

(b) Show that f is not uniformly continuous on $(-1, 0) \cup (0, 1)$.

3. Consider the sequence of functions

$$f_k(x) = k^2 x^2 e^{-k^2 x}.$$

Prove that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on $[0, \infty)$.

4. Prove that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ and $f'(x) \geq 1$ for all $x \neq 0$. [*Hint:* Use the Mean Value Theorem.]

5. (a) State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$.

(b) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and monotonically increasing, with $f(0) = 0$, $f(1/2) = 1$, and $f(1) = 2$. Prove that

$$\int_0^1 f(x) \, dx > 1/2.$$

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.