Real Analysis Qualifying Exam, September 14, 2019

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. (a) Prove that the sequence defined by $x_1 = 0$ and $x_{n+1} = \dfrac{x_n^2 + 2}{3}$ converges.

(b) Explicitly compute the limit of the sequence in part (a).

2. Show that the sequence of functions

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$$

converges pointwise to $f(x) = 0$ on $[0, 1]$, but does not converge uniformly.

3. Consider the function $f(x) = \sum_{k=1}^{\infty}(1 - \cos(x/k))$.

You may use without proof the following inequalities in this problem:

$$|\sin t| \leq |t|, \quad |1 - \cos t| \leq \frac{t^2}{2}, \quad t \in \mathbb{R}.$$

(a) Prove that the series for f converges uniformly on every interval of the form $[-M, M]$ in $\mathbb{R}$.

(b) Prove that f is differentiable on $\mathbb{R}$.

4. (a) State the Mean Value Theorem.

(b) Use the Mean Value Theorem to prove that $|\tan x| \geq |x|$ for all $x \in (-\pi/2, \pi/2)$.

5. (a) State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$.

(b) Let a_n be a positive sequence of real numbers converging to 0 and let $B = \{b_1, b_2, b_3, \ldots\}$ be a countably infinite subset of $[0, 1]$. Consider the function f on $[0, 1]$ defined by

$$f(x) = \begin{cases} a_n, & x = b_n \\ 0, & x \notin B \end{cases}.$$

Use your definition from (a) to prove that f is Riemann integrable on $[0, 1]$.

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.