Real Analysis Qualifying Exam, September 16, 2018

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Consider the sequence

$$a_n = \frac{2 + 2^{\frac{1}{2}} + 2^{\frac{1}{3}} + \ ... \ + 2^{\frac{1}{n}}}{n}.$$

(a) Show that the sequence $\{a_n\}$ is decreasing.

(b) Show that the sequence $\{a_n\}$ converges.

(c) Show that $\lim_{n \to \infty} a_n = 1$.

2. (a) Define what it means for $f : A \subseteq \mathbb{R} \to \mathbb{R}$ to be uniformly continuous.

(b) Use the definition to show that $f(x) = 1/x$ is uniformly continuous on $(1, 2)$.

(c) Show that $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

3. Consider the function

$$f(x) = \sum_{k=0}^{\infty} e^{-kx} \cos kx.$$

(a) Prove that the series converges uniformly on $[a, \infty)$ for any $a > 0$.

(b) Prove that f is a continuous function on $(0, \infty)$.

4. Suppose that f is differentiable on $\mathbb{R}$ and that $f'(x) \leq 4$ for all $x \in \mathbb{R}$. Prove that there is at most one point $x > 2$ such that $f(x) = x^2$.

5. (a) State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$.

(b) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function with the property that f is Riemann integrable on $[a, c]$ for all $a < c < b$. Use the definition of Riemann integrability to show that f is Riemann integrable on $[a, b]$.

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.