Real Analysis Qualifying Exam, September 10, 2017

Instructions: This exam consists of 5 questions. Each question is worth 5 points, giving a grand total of 25 points possible. Please present all of your work in a clear and concise manner and answer each question as completely as possible. Unsupported work will receive no credit and partially completed work may receive partial credit. Good luck!

1. Prove that the sequence $\{a_n\}$, where

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}},$$

converges and compute its limit.

2. The following proposition is both imprecise and incorrect:

If f is continuous and $\{x_n\}$ is convergent, then $\{f(x_n)\}$ is also convergent.

(a) Give an example to show why this proposition is false. (For a sufficiently simple example, no justification is necessary.)

(b) State a precise and correct version of this proposition, and then prove it.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous and twice differentiable on (a, b). Suppose also that the line segment joining the points $(a, f(a))$ and $(b, f(b))$ meets the graph of f at a point $(c, f(c))$, where $a < c < b$. Prove that there exists $d \in (a, b)$ such that $f''(d) = 0$.

4. Consider the series

$$\sum_{n=1}^{\infty} e^{-nx^2} \sin(nx).$$

(a) Prove that this series converges uniformly on $[a, \infty)$, for each $a > 0$.

(b) Does the series converge uniformly on $[0, \infty)$? Justify your answer.

5. (a) State the definition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$.

(b) Use your definition from (a) to prove that

$$f(x) = \begin{cases} 1, & x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

is integrable on $[0, 1]$ and compute the value of the integral $\int_0^1 f(x)\, dx$.

Note: If you choose to work with a definition of Riemann integrability different than that stated in part (a), please provide the alternate definition.