Algebra Qualifying Exam

June 9, 2018

1. Let G be a group and $a \in G$ be an element. Let $n \in \mathbf{N}$ be the smallest positive number such that $a^n = e$, where e is the identity element. Show that the set

   $$\{e, a, a^2, \ldots, a^{n-1}\}$$

   contains no repetitions.

2. Let G be a finite group and $H, K \trianglelefteq G$ be normal subgroups of relatively prime order. Prove that G is isomorphic to a subgroup of $G/H \times G/K$.

3. Prove that if $\phi : R \to S$ is a surjective ring homomorphism between commutative rings with unity, then $\phi(1_R) = 1_S$.

4. Let $V \subset \mathbf{R}^5$ be the subspace defined by the equation

   $$x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_5 = 0.$$

   a) Find (with justification) a basis for V.

   b) Find (with justification) a basis for $V^\perp$, the subspace of $\mathbf{R}^5$ orthogonal to V under the usual dot product.

5. Suppose V is a finite-dimensional real vector space and $T : V \to V$ is a linear transformation. Prove that T has at most $\dim(\operatorname{range} T)$ distinct nonzero eigenvalues.