Algebra Qualifying Exam

September 12, 2020

1. Let $A = \begin{bmatrix} 6 & -2 & -1 \\ 10 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

   a) Find bases for the eigenspaces of A.

   b) Determine if A is diagonalizable. If so, give an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. If not, explain why not.

2. Let G be the additive group $\mathbf{Z}_{2020}$ and let $H \subseteq G$ be the subset consisting of those elements with order dividing 20.

   a) Prove H is a subgroup of G.

   b) Find an explicit generator for H and determine its order.

3. Let G be a finite group and $\mathrm{Z}(G)$ denote its center.

   a) Prove that if $G/\mathrm{Z}(G)$ is cyclic, then G is abelian.

   b) Prove that if G is nonabelian, then $|\mathrm{Z}(G)| \leq \frac{1}{4}|G|$.

4. Let R be a commutative ring with 1. We say an element $n \in R$ is **nilpotent** if there exists a number $k \in \mathbf{N}$ such that $n^k = 0$.

   a) Show that if n is nilpotent, then $1 - n$ is a unit.

   b) Give an example of a commutative ring with 1 that has no nonzero nilpotent elements, but is not an integral domain.

5. Let R be a ring with 1 and suppose $e \in R$ is **idempotent**, i.e., satisfies $e^2 = e$.

   a) Prove that $1 - e$ is also idempotent.

   b) Suppose $e \neq 0, 1$. Show that Re and $R(1-e)$ are proper left ideals of R.

   c) Prove there is an isomorphism $R \cong Re \times R(1-e)$.