Algebra Qualifying Exam

September 15, 2019

1. Let P_3 be the real vector space of all real polynomials of degree three or less. Define $L : P_3 \to P_3$ by $L(p(x)) = p(x) + p(-x)$.
   a) Prove L is a linear transformation.
   b) Find a basis for the null space of L.
   c) Compute the dimension of the image of L.

2. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that rotates counterclockwise around the z-axis by $\frac{2\pi}{3}$.
   a) Write the matrix for T with respect to the standard basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.
   b) Write the matrix for T with respect to the basis $\left\{ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.
   c) Determine all (complex) eigenvalues of T.
   d) Is T diagonalizable over $\mathbf{C}$? Justify your answer.

3. Suppose G is a cyclic group of order n, and $t \in G$ is a generator.
   a) Give a positive integer d such that $t^{-1} = t^d$.
   b) Let c be an integer and let $m = \gcd(n, c)$. Prove that the order of t^c is $\frac{n}{m}$.

4. Suppose G is a group, H and K are normal subgroups of G, and $H \leq K$.
   a) Define a group homomorphism from K to G/H.
   b) Compute the kernel of the homomorphism in (a), and apply the First Isomorphism Theorem.

5. Let $I \subseteq \mathbf{Z}[x]$ denote the set of all polynomials with even constant term.
   a) Prove that I is an ideal.
   b) Prove that I is not a *principal* ideal.