Algebra Qualifying Exam

September 15, 2018

1. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2z-x \\ y+2z \end{bmatrix}$.

   a) Find the matrix that represents T with respect to the standard basis for $\mathbf{R}^3$.

   b) Find a basis for the kernel of T.

   c) Determine the rank of T.

2. Suppose G is a group, $H \leq G$ a subgroup, and $a, b \in G$. Prove that the following are equivalent:

   a) $aH = bH$

   b) $b \in aH$

   c) $b^{-1}a \in H$

3. Let G be a group and $H, K \trianglelefteq G$ be normal subgroups with $H \cap K = \{e\}$. Show that each element in H commutes with every element in K.

4. Let R be a commutative ring with unity.

   a) Define what it means for an element in R to be **prime**, and also what it means for an element to be **irreducible**.

   b) Prove that if R is an integral domain, then every prime element is irreducible.

5. Suppose A is a real $n \times n$ matrix that satisfies $A^2\mathbf{v} = 2A\mathbf{v}$ for every $\mathbf{v} \in \mathbf{R}^n$.

   a) Show that the only possible eigenvalues of A are 0 and 2.

   b) For each $\lambda \in \mathbf{R}$, let E_λ denote the λ-eigenspace of A, i.e., $E_\lambda = \{\mathbf{v} \in \mathbf{R}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}$. Prove that $\mathbf{R}^n = E_0 \oplus E_2$. (*Hint:* For every vector $\mathbf{v}$ one can write $\mathbf{v} = (\mathbf{v} - \frac{1}{2}A\mathbf{v}) + \frac{1}{2}A\mathbf{v}$.)