



## **Exploring the Relationship between Dynamics and Stability**

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## Exploring the Relationship between Dynamics and Stability

Structural engineering students have long struggled, both in the undergraduate and graduate level, with structural dynamics and stability. The two topics are generally taught separately but with a similar approach: first using a differential equation formulation; and then, as the problems become increasingly complex, using a matrix-based eigen-analysis approach. Given that many students struggle with math, it's no wonder that student comprehension in both topics is often lacking.

This paper describes the development, implementation, and assessment of a simple laboratory exercise in which students explore the interconnectivity between structural dynamics and structural stability. The objective of the laboratory is to demonstrate the effect of axial load on vibration frequency i.e. as the buckling load is approached, the natural frequency drops to zero. Mathematically, the similarity between buckling and stability in the differential equation and eigen-analysis approach is obvious to the faculty member. However in the students' minds, these topics bear little to no resemblance to each other; after all, they were taught in different courses. The goal of this paper is to show that student comprehension of this complex interaction between structural stability and dynamics can be improved using a simple, hands-on laboratory exercise.

Prior to the activity, the students' understanding and interest in beam vibration and column buckling was assessed through a quiz-for-grade and found to be typical for senior undergraduates; i.e. weak. During the laboratory, it rapidly becomes clear that the students' interest level has risen. Without any prompting, the students immediately start to explore the test setup by pressing down on the columns by hand and observing the shape associated with the end restraints. Separately, they observe the accelerometer output and take interest seeing the effects of plucking the column (like a guitar string) and observing the motion on the computer screen. It is when the column loading and the plucking action are combined in a more formal laboratory exercise that the students visibly become intrigued. Observing the drop off in frequency as the column is loaded leaves them with no doubt that the column flexural stiffness is affected by the presence of axial loads. In other words, the code-based P-delta effect has been proven. Assessments taken after the exercise confirm the obvious. Not only has the students' interest in the buckling and stability been greatly enriched, their ability to apply the concepts and relate them together has demonstrably improved.

### Student Background

The students involved in this exercise were senior undergraduates and first-year graduate students in an introductory nonlinear analysis course. They had all previously taken course in structural dynamics and matrix structural analysis. While the current course spanned over two 10-week quarters, the exercise was performed after 9 weeks of instruction. The goal of these 9 weeks was to lead the students through the theory of structural stability and in particular the P-Delta effect. The course started with the buckling of rigid links connected by springs, using equilibrium and energy-based approaches. Next, the students studied the buckling of flexible beams and columns using a differential equation approach. Finally, the students tackled the matrix formulation for frame structures using stability functions (Chen [1]) to account for the

reduction of flexural stiffness in the presence of axial compression, i.e. the P-Delta effect. Through the use of homework, quizzes and exams, the students had demonstrated their supposed mastery of the subject material. That is to say they could apply both a differential equation approach and a matrix-based eigen-analysis approach to solve the given problems. The purpose of the laboratory exercise was to ensure that the students actually understood the P-Delta effect on a conceptual level and not just the mathematics behind it.

### Pre-test Assessment

To assess the students' conceptual understanding of the P-Delta effect, they were given a non-numerical quiz in which they were given a series of similar structures and asked if the deflections or natural frequencies of the structures were identical. The structures differed only in the axial forces that were present in the flexural members. Since the students had spent the previous 9 weeks calculating the precise deflections and critical loads in such structures, one might think that the quiz would be trivial. This of course was not the case.

Figure 1 shows one such question from the quiz. In each of the three structures, an identical cantilever supports a heavy mass represented by a square. The students were asked to comment on the natural frequency of the system (or the displacements due only to the load  $H$  for those that felt more comfortable with a static load). The cantilever on the left is in compression, the one on the right is in tension, and the middle cantilever has no axial load. The compression in the left-hand column causes a reduction in the flexural stiffness of the column (the P-Delta effect); thus the natural frequency of the left column will be lower than the middle structure. Conversely, the tension in the column on the right causes flexural stiffening and a higher natural frequency.

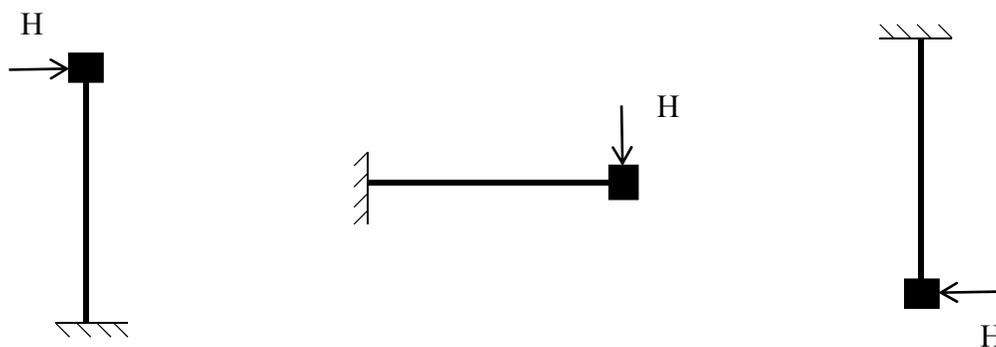


Figure 1: Pre-Assessment Test

Had the quiz involved numbers, calculations, matrices and/or differential equations, the students no doubt would have been able to solve it. However faced with only a conceptual question, only 5 out of 19 students correctly solved the problem by identifying the P-Delta effect. The vast majority of the class failed to even think about the effect of axial force on flexural stiffness. And yet they were in a course solely devoted to the concept. Clearly this is not acceptable.

Many authors have found (see McDaniel [2] for example) in the past that simple physical experiments are often useful for driving home the fundamentals of a problem. Eliminating the calculations and letting the students rapidly explore actual phenomena at hand yields great

benefits. Unfortunately, P-Delta effects are either subtle and hence difficult to observe, or they are quite overt and result in a rapidly collapsing structure from which is again difficult to explore. However, as a structure takes on increasing axial loads its flexural stiffness decreases and its natural frequency noticeably decreases. In the extreme, the natural frequency drops to zero at the point of buckling. Thus by placing accelerometers on a given structure and in real time observing the change in natural frequency as the loads are changed, the students can quickly determine and explore P-Delta effects using structural dynamics.

### Theory

The relationship between free vibration and elastic stability is generally well known in academia. Housner and his students [3] at Cal Tech in the 1940's and 1950's experimentally determined the critical load of a member by means of linearly correlating the square of the frequency and the load; the extrapolated regression line at zero frequency represents the critical buckling load of the member. On the theoretical side, the most acknowledged achievements are traced back to Timoshenko [4] who used differential equations to show that as the loading of a column approached Euler buckling, the frequency of the first mode approaches zero. Recently Carpinteri [5] presented a solution using potential energy that is well-tailored to student understanding. His derivation uses a single degree-of-freedom system similar to that shown in Figure 2.

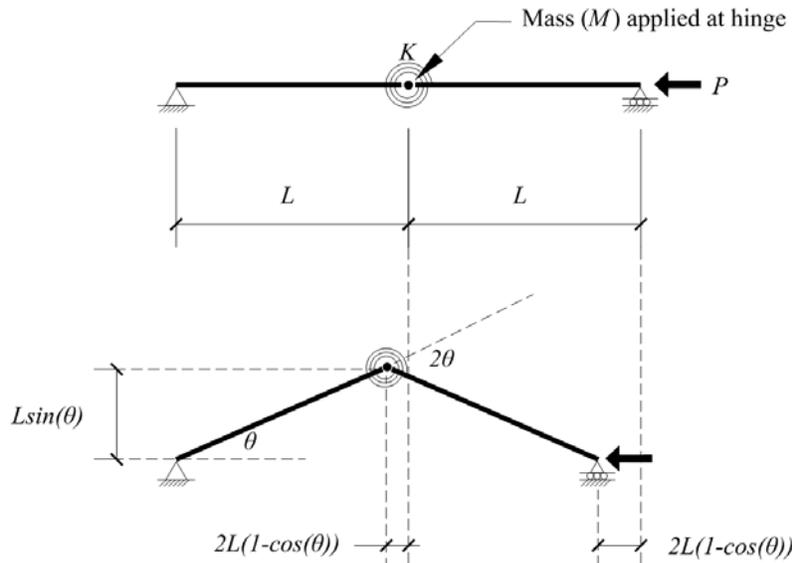


Figure 2: SDOF system

The total potential energy of the system in Figure 2 is:

$$\Pi = \frac{1}{2}k(2\theta)^2 - P2L(1 - \cos \theta) \quad (\text{eq. 1})$$

Because the stated problem is dynamic, inertia must be accounted for. The system has velocities in two directions, horizontal and vertical. These velocities are related through the rotation of the bars and must be added together as:

$$K_e = \frac{1}{2}M \left( \frac{d(L \sin \theta)}{dt} \right)^2 + \frac{1}{2}M \left( \frac{d(L(1-\cos \theta))}{dt} \right)^2 \quad (\text{eq. 2})$$

which simplifies to:

$$K_e = \frac{1}{2}ML^2\dot{\theta}^2 \quad (\text{eq. 3})$$

To relate the kinetic energy of the system to the potential energy, Lagrange's equation can be utilized

$$\frac{\partial}{\partial t} \left( \frac{\partial K_e}{\partial \dot{\theta}} \right) - \frac{\partial K_e}{\partial \theta} = - \frac{\partial \Pi}{\partial \theta} \quad (\text{eq. 4})$$

Substituting into Lagrange's equation yields

$$ML^2\ddot{\theta} = -(4k\theta - 2Pl \sin \theta) \quad (\text{eq. 5})$$

A small displacement approximation is made which simplifies to

$$ML^2\ddot{\theta} = -(4k\theta - 2Pl\theta) \quad (\text{eq. 6})$$

Substituting in the general equation

$$\theta = \theta_0 e^{i\omega t} \quad (\text{eq. 7})$$

yields

$$\theta(4k - 2PL - \omega^2 ML^2) = 0 \quad (\text{eq. 8})$$

Thus

$$P = \frac{2K}{L} - \frac{ML}{2}\omega^2 \quad (\text{eq. 9})$$

Assuming the mass of the system is small compared to the axial load at buckling yields

$$P_{cr} = \frac{2K}{L} \quad (\text{eq. 10})$$

When the system has no axial load, the system's natural frequency is

$$\omega_n = \sqrt{\frac{4K}{mL^2}} \quad (\text{eq. 11})$$

Combining the results yields the following interaction equation

$$\frac{P}{P_{cr}} + \frac{w^2}{w_n^2} = 1 \quad (\text{eq. 12})$$

This shows that as the axial load of the system approaches the critical buckling load, the square of the natural frequency reaches zero, and the system will buckle; as axial load increases, the stiffness of the system lowers. By plotting this relationship is easily recognized (see figure 3).

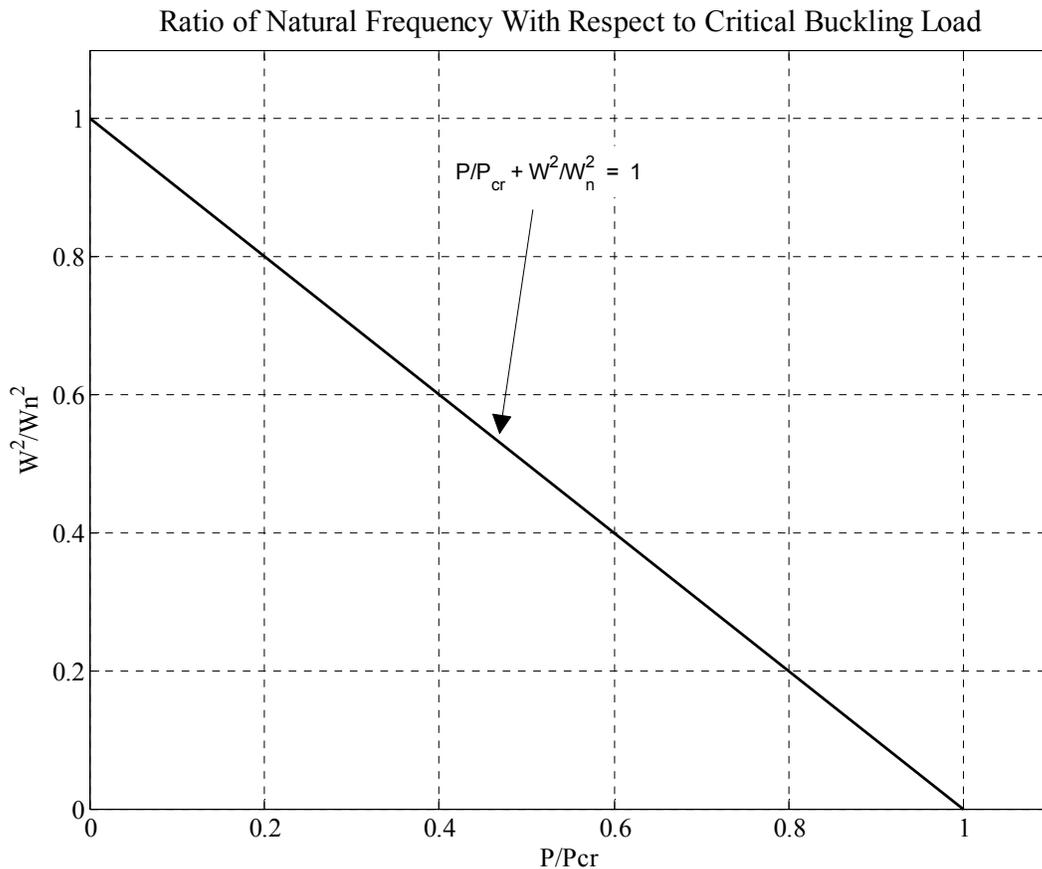


Figure 3: The relationship between axial load ( $P/P_{cr}$ ) and natural frequency ( $w^2/w_n^2$ )

It is important to mention that Figure Figure 3: shows a linear relationship. This relationship holds true for a one degree-of-freedom where the mode shape for the buckled system is the same as the mode shape for the free vibration of the first mode. For a flexible system, the relationship is not linear; however the non-linearity is small and can well represented by a linear equation. The importance of this derivation is that it shows that as axial load increases, the natural frequency of the system decreases. As the system approaches buckling, the natural frequency goes to zero. A contrary analogy can be used for a tight wire or guitar string. As a guitar string

is tightened, the audible frequency goes higher.

## Experiment

The above theoretical derivation yields a very simple yet informative experiment. A small cantilever column with a mass on top (as shown in figure 1, the pre-test assessment) was constructed from an old band saw blade and some plywood (figure 4). An accelerometer is attached low on the cantilever so as to not add much mass to the system. Through standard data acquisition hardware and software, the natural frequency of the cantilever is shown on the computer screen in real-time. The cantilever is first held upright (left-hand photo), then laid on its side (center photo), and then finally hung down from the table top (right-hand photo). In each position the natural frequency is read off the computer screen.

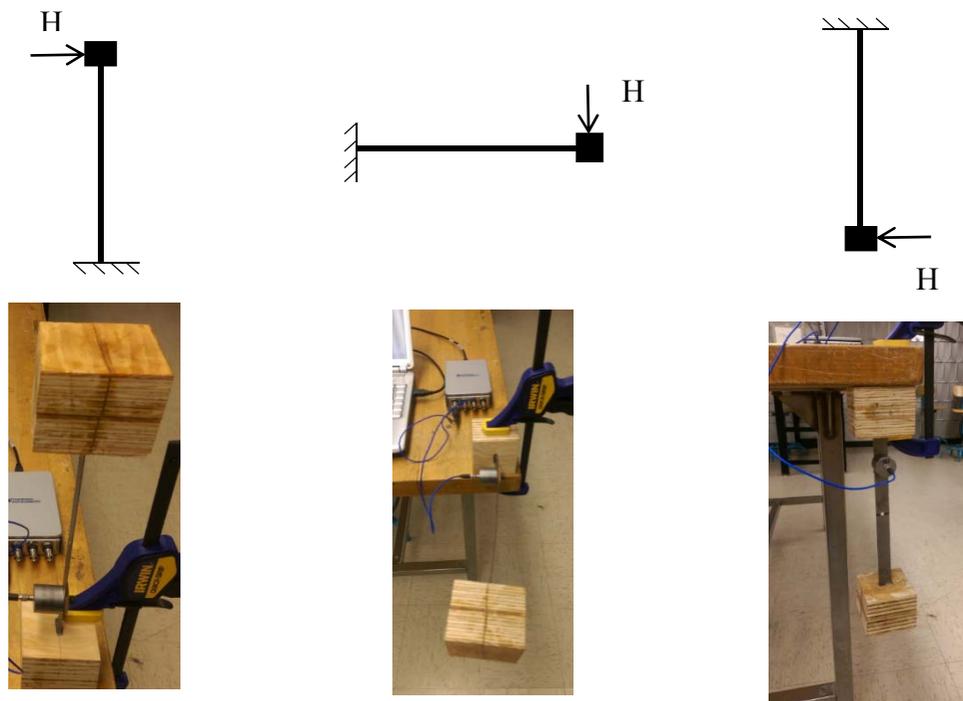


Figure 4: Experimental setup

By weighing the wood block ahead of time, the three results can be plotted (see figure 5). The center data point is from the cantilever laying horizontal (zero axial load). The right-hand data point comes from the upright cantilever (positive load in compression, with a slightly lower frequency). The left-hand data point is from the cantilever suspended below the table (negative load in compression, with a slightly higher frequency). As detailed in the theory, a line can be extended through the three points. The extension of the line through the horizontal axis represents a natural frequency of zero which corresponds to the predicted buckling load (in this case 3.87 lbs).

The students were noticeably interested in the experiment. In post-test evaluation, students indicated that the experiment was “the most memorable thing in the course”. Even though the

theory absolutely predicted the results, the students wanted to see the experiment repeated. It was very clear, compression softened the flexural stiffness and tension provided stiffening -- exactly as P-Delta theory predicts. After observing the increase in frequency from the upright position to the horizontal position, many students wanted to change their quiz answers. They now anticipated that putting the column in tension by hanging the cantilever from the desk would increase the stiffness, and hence the frequency, further. The experiment seemed to be a success. The noise-level in the classroom was to the point of disruptive. The students were observed arguing amongst themselves over the application of the P-Delta effect in current and past homework problems.

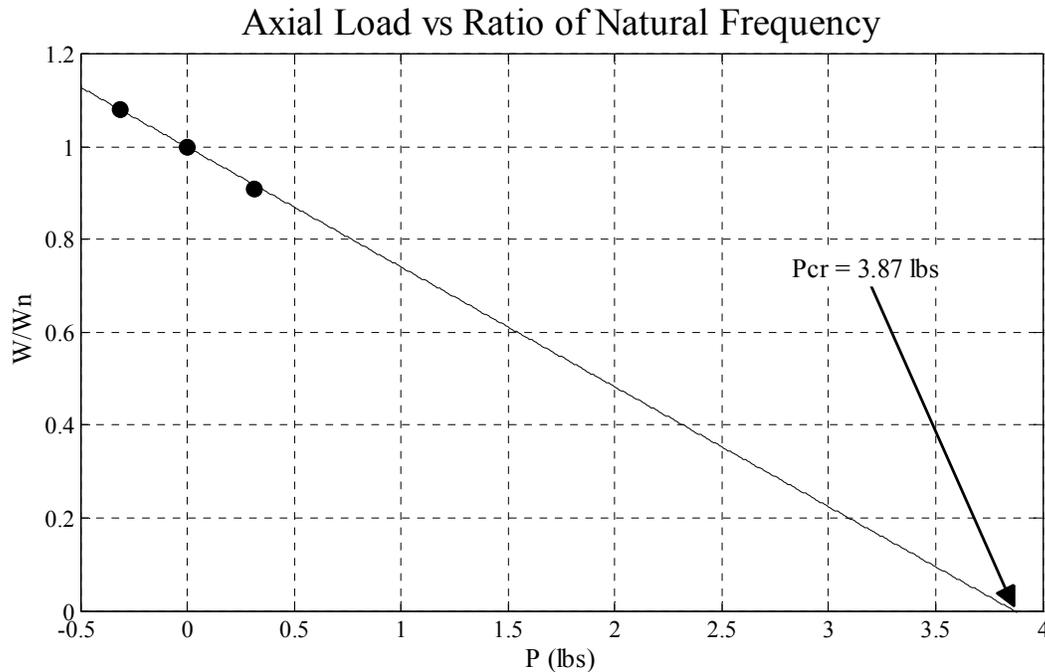


Figure 5: Experimental Results, Axial Load (P) vs. Natural Frequency ( $w^2/w_n^2$ )

### Post-test Assessment

One week after the laboratory exercise, the students were given an exam that contained another conceptual question (shown below in Figure 6). The students were asked if the lateral deflections of the two statically determinate systems were identical if axial deformations were ignored. The only difference between the two systems is that in the structure on the left, the beam is in compression while on the right, the beam has no axial load. In both cases, the loads on the columns are identical. Since the lateral resistance of the frame involves bending the beam, the left-hand structure is more flexible due to the P-Delta effect.

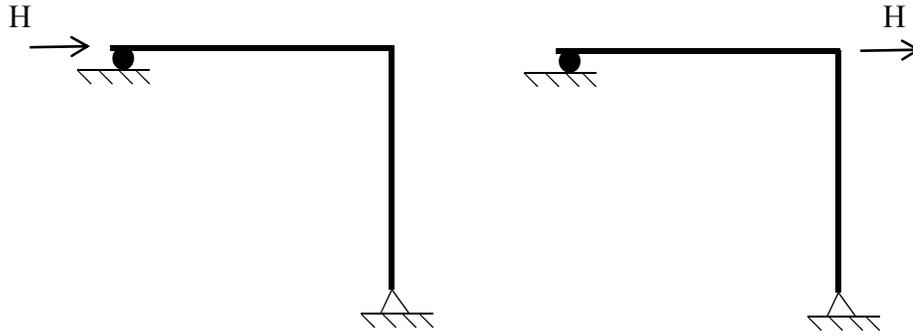


Figure 6: Post-test Assessment question

While the pre and post assessment questions appear different, they focused on the same issue. Namely, can the students recognize situations in which the axial forces affect flexural stiffness? Of the 19 students in the class, only 3 failed to recognize the P-Delta effect. In the pre-test assessment, 14 students failed to recognize it. The physical experiment greatly improved the students' conceptual understanding of the P-Delta effect. Of the 16 students who got the post-test assessment correct, curiously 5 students recognized the P-Delta effect in the left-hand beam but also thought the beam on the right-hand structure was in tension. Perhaps a little brush up in basic statics is in order.

## Conclusion

The introduction of a simple experiment in which the relationship between stability and vibration frequency was explored, drastically improved the students' conceptual understanding of the P-Delta effect. Prior to the experiment, only 26% of students could identify a reduction in flexural stiffness due to axial compression. After the experiment, 84% of students correctly predicted the P-Delta effect. The experiment itself is simple. It consists of small cantilever and an accelerometer. When held upright the natural frequency drops due to the compression in the cantilever. When held upside down the opposite occurs. The students found this to be fascinating and memorable display of the P-Delta effect.

## References

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