Cost Minimization and Cost Curves

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Sections: 3.1a, 3.2a-b, 4.1

Agenda

- The Cost Function and General Cost Minimization
- Cost Minimization with One Variable Input
- Deriving the Average Cost and Marginal Cost for One Input and One Output
- Cost Minimization with Two Variable Inputs

Cost Function

- A cost function is a function that maps a set of inputs into a cost.
- In the short-run, the cost function incorporates both fixed and variable costs.
- In the long-run, all costs are considered variable.

Cost Function Cont.

- The cost function can be represented as the following:
  - $C = c(x_1, x_2, \ldots, x_n) = w_1 * x_1 + w_2 * x_2 + \ldots + w_n * x_n$
  - Where w_i is the price of input i, x_i, for $i = 1, 2, \ldots, n$
  - Where C is some level of cost and c(•) is a function

Cost Function Cont.

- When there are fixed costs, the cost function can be represented as the following:
  - $C = c(x_1, x_2 | x_3, \ldots, x_n) = w_1 * x_1 + w_2 * x_2 + TFC$
  - Where w_i is the price of the variable input i, x_i, for i = 1 and 2
  - Where w_i is the price of the fixed input i, x_i, for i = 3, 4, …, n
  - Where $TFC = w_3 * x_3 + \ldots + w_n * x_n$
    - When inputs are fixed, they can be lumped into one value which we usually denote as TFC
    - $x_3, \ldots, x_n$ are held to some constant values that do not change

Cost Function Cont.

- Suppose we have the following cost function:
  - $C = c(x_1, x_2, x_3) = 5*x_1 + 9*x_2 + 14*x_3$
  - If x3 was held constant at 4, then the cost function can be written as:
  - $C = c(x_1, x_2 | 4) = 5*x_1 + 9*x_2 + 56$
    - Where TFC in this case is 56

Cost Function Cont.

- The cost function is usually meaningless unless you have some constraint that bounds it, i.e., minimum costs occur when all the inputs are equal to zero.

Standard Cost Minimization Model

- Assume that the general production function can be represented as $y = f(x_1, x_2, \ldots, x_n)$.

$$\underset{w.r.t.\, x_1, x_2, \ldots, x_n}{Min} \quad w_1 x_1 + w_2 x_2 + \ldots + w_n x_n$$

$$\text{subject to} : y = f(x_1, x_2, \ldots, x_n)$$

Cost Minimization with One Variable Input

- Assume that we have one variable input (x) which costs w. Let TFC be the total fixed costs.
- Assume that the general production function can be represented as y = f(x).

$$\underset{w.r.t.\,x}{Min}\; wx + TFC$$

$$\text{subject to}: y = f(x)$$

Lagrangian Solution for One Variable Input Model

- $\Gamma(x) = wx + TFC + \lambda\big(y - f(x)\big)$
- FOC =>
- $\frac{\partial \Gamma(x)}{\partial x} = w - \lambda \frac{\partial f(x)}{\partial x} = 0$
- $\frac{\partial \Gamma(x)}{\partial \lambda} = y - f(x) = 0$
- $\lambda = \frac{w}{\frac{\partial f(x)}{\partial x}}$ = MC

Cost Minimization with One Variable Input Cont.

- In the one input, one output world, the solution to the minimization problem is trivial.
  - By selecting a particular output y, you are dictating the level of input x.
  - The key is to choose the most efficient input to obtain the output.

Example of Cost Minimization

- Suppose that you have the following production function:
  - $y = f(x) = 6x - x^2$
- You also know that the price of the input is $10 and the total fixed cost is 45.

$$\underset{w.r.t.\,x}{Min}\ 10x + 45$$

$$\text{subject to}: y = f(x) = 6x - x^2$$

Example of Cost Minimization Cont.

- Since there is only one input and one output, the problem can be solved by finding the most efficient input level to obtain the output.

$$y = 6x - x^2$$

$$\Rightarrow x^2 - 6x + y = 0$$

$$\Rightarrow x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(y)}}{2}$$

$$\Rightarrow x = \frac{6 \pm \sqrt{36 - 4y}}{2}$$

$$x = \frac{6 \pm \sqrt{4}\sqrt{9 - y}}{2}$$

$$x = 3 \pm \sqrt{9 - y}$$

Example of Cost Minimization Cont.

- Given the previous, we must decide whether to use the positive or negative sign.
  - This is where economic intuition comes in.
  - The one that makes economic sense is the following:

$$x = 3 - \sqrt{9 - y}$$

Cost Function and Cost Curves

- There are many tools that can be used to understand the cost function:
  - Average Variable Cost (AVC)
  - Average Fixed Cost (AFC)
  - Average Cost (ATC)
  - Marginal Cost (MC)

Average Variable Cost

- Average variable cost is defined as the cost function without the fixed costs divided by the output function.

$$AVC = \frac{wx}{y}$$

$$AVC = \frac{wx}{f(x)}$$

$$AVC = \frac{w}{APP}$$

Average Fixed Cost

- Average fixed cost is defined as the cost function without the variable costs divided by the output function.

$$AFC = \frac{TFC}{y}$$

$$AFC = \frac{TFC}{f(x)}$$

Average Total Cost

- Average total cost is defined as the cost function divided by the output function.
- It is also the summation of the average fixed cost and average variable cost.

$$ATC = \frac{wx + TFC}{y}$$

$$ATC = \frac{wx}{y} + \frac{TFC}{y}$$

$$ATC = \frac{wx}{f(x)} + \frac{TFC}{f(x)}$$

Marginal Cost

- Marginal cost is defined as the derivative of the cost function with respect to the output.
- To obtain MC, you must substitute the production function into the cost function and differentiate with respect to output.

$$MC = \frac{dTC(y)}{dy} = \frac{dTVC(y)}{dy}$$

$$TC(y) = w * f^{-1}(y) + TFC$$

$$TVC(y) = w * f^{-1}(y)$$

$$f^{-1}(y) \text{ is inverse of } y = f(x)$$

$$MC = \frac{w}{MPP}$$

Example of Finding Marginal Cost

- Using the production function $y = f(x) = 6x - x^2$, and a price of 10, find the MC by differentiating with respect to y.
- To solve this problem, you need to solve the production function for x and plug it into the cost function.
  - This gives you a cost function that is a function of y.

Example of Finding Marginal Cost Cont.

$$y = 6x - x^2$$

$$\Rightarrow x = 3 - \sqrt{9 - y}$$

Plugging this into the cost function gives:

$$\mathrm{C} = \mathrm{c(y)} = 10(3 - \sqrt{9 - y})$$

$$\Rightarrow \mathrm{C} = \mathrm{c(y)} = 30 - 10\sqrt{9 - y}$$

$$MC = \frac{dc}{dy} = 0 - \left(\frac{1}{2} * 10(9 - y)^{-\frac{1}{2}}(-1) \right)$$

$$MC = \frac{dc}{dy} = \left(\frac{1}{2} * 10(9 - y)^{-\frac{1}{2}} \right)$$

$$MC = \frac{dc}{dy} = \left(\frac{5}{\sqrt{9 - y}} \right)$$

Notes on Costs

- MC will meet AVC and ATC from below at the corresponding minimum point of each.
  - Why?
- As output increases AFC goes to zero.
- As output increases, AVC and ATC get closer to each other.

Production and Cost Relationships Summary

- Cost curves are derived from the physical production process.
- The two major relationships between the cost curves and the production curves:
  - AVC = w/APP
  - MC = w/MPP

Product Curve Relationships Cont.

$$AVC = \frac{wx}{f(x)}$$

$$\frac{dAVC}{dx} = \frac{wf(x) - wxf'(x)}{[f(x)]^2} \overset{>}{\underset{<}{=}} 0$$

$$\Rightarrow wf(x) - wxf'(x) \overset{>}{\underset{<}{=}} 0$$

$$\Rightarrow f(x) - xf'(x) \overset{>}{\underset{<}{=}} 0$$

$$\Rightarrow f(x) \overset{>}{\underset{<}{=}} xf'(x)$$

Product Curve Relationships Cont.

$$\Rightarrow f(x) \underset{<}{\overset{>}{=}} xf'(x)$$

$$\Rightarrow \frac{f(x)}{x} \underset{<}{\overset{>}{=}} f'(x)$$

$$\Rightarrow APP \underset{<}{\overset{>}{=}} MPP$$

$$\Rightarrow \frac{1}{MPP} \underset{<}{\overset{>}{=}} \frac{1}{APP}$$

$$\Rightarrow \frac{w}{MPP} \underset{<}{\overset{>}{=}} \frac{w}{APP}$$

$$\Rightarrow MC \underset{<}{\overset{>}{=}} AVC$$

Product Curve Relationships

- When MPP>APP, APP is increasing.
  - => MC<AVC, then AVC is decreasing.
- When MPP=APP, APP is at a maximum.
  - => MC=AVC, then AVC is at a minimum.
- When MPP<APP, APP is decreasing.
  - => MC>AVC, then AVC is increasing.

Example of Examining the Relationship Between MC and AVC

- Given that the production function $y = f(x) = 6x - x^2$, and a price of 10, find the input(s) where AVC is greater than, equal to, and less than MC.
- To solve this, examine the following situations:
  - AVC = MC
  - AVC > MC
  - AVC < MC

Example of Examining the Relationship Between MC and AVC Cont.

$$AVC = \frac{wx}{f(x)} = \frac{10x}{6x - x^2} = \frac{10}{6 - x}$$

$$MC = \frac{w}{MPP} = \frac{10}{6 - 2x}$$

$$AVC = MC$$

$$\Rightarrow \frac{10}{6 - x} = \frac{10}{6 - 2x}$$

$$\Rightarrow 6 - x = 6 - 2x$$

$$\Rightarrow 2x = 0$$

$$\Rightarrow x = 0$$

Example of Examining the Relationship Between MC and AVC Cont.

$$AVC = \frac{wx}{f(x)} = \frac{10x}{6x - x^2} = \frac{10}{6 - x}$$

$$MC = \frac{w}{MPP} = \frac{10}{6 - 2x}$$

$$AVC > MC$$

$$\Rightarrow \frac{10}{6 - x} > \frac{10}{6 - 2x}$$

$$\Rightarrow 6 - 2x > 6 - x$$

$$\Rightarrow -2x > -x$$

$$\Rightarrow 0 > x$$

Example of Examining the Relationship Between MC and AVC Cont.

$$AVC = \frac{wx}{f(x)} = \frac{10x}{6x - x^2} = \frac{10}{6 - x}$$

$$MC = \frac{w}{MPP} = \frac{10}{6 - 2x}$$

$$AVC < MC$$

$$\Rightarrow \frac{10}{6 - x} < \frac{10}{6 - 2x}$$

$$\Rightarrow 6 - x > 6 - 2x$$

$$\Rightarrow x > 0$$

Review of the Iso-Cost Line

- The iso-cost line is a graphical representation of the cost function with two inputs where the total cost C is held to some fixed level.
  - $C = c(x_1,x_2)=w_1x_1 + w_2x_2$

Finding the Slope of the Iso-Cost Line

$$C = w_1 x_1 + w_2 x_2$$

$$\Rightarrow w_2 x_2 = C - w_1 x_1$$

$$\Rightarrow \frac{w_2 x_2}{w_2} = \frac{C}{w_2} - \frac{w_1 x_1}{w_2}$$

$$\Rightarrow x_2 = \frac{C}{w_2} - \frac{w_1 x_1}{w_2}$$

$$\frac{dx_2}{dx_1} = -\frac{w_1}{w_2}$$

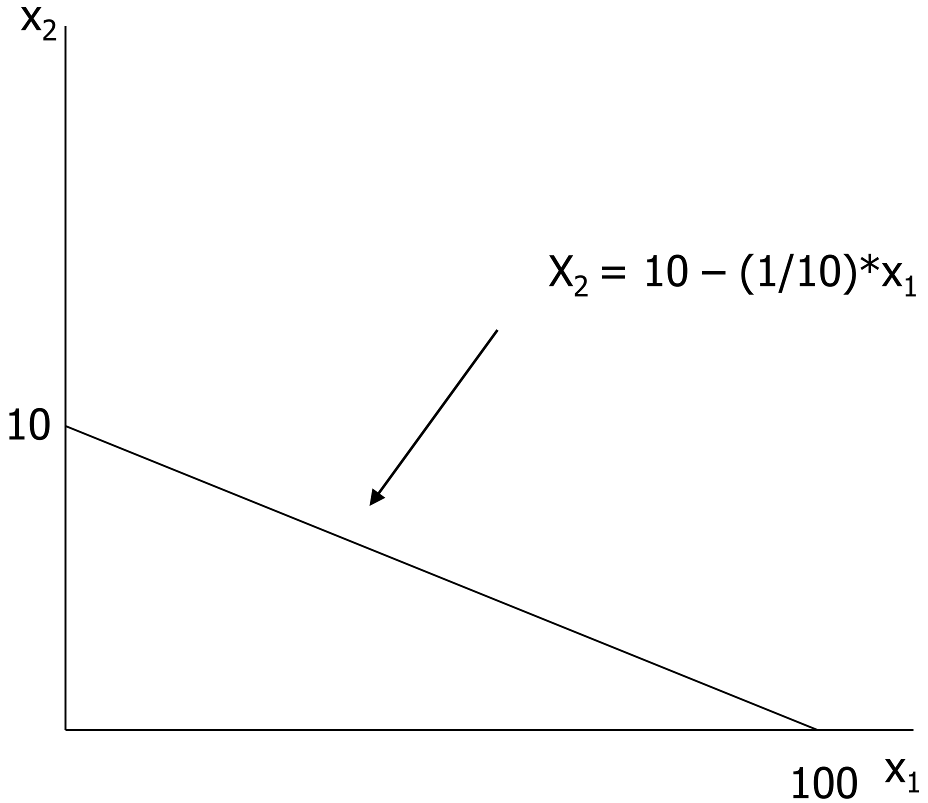
Example of Iso-Cost Line

- Suppose you had $1000 to spend on the production of lettuce.
- To produce lettuce, you need two inputs labor and machinery.
  - Labor costs you $10 per unit, while machinery costs $100 per unit.

Example of Iso-Cost Line Cont.

- Given the information above we have the following cost function:
  - C = c(labor, machinery) = \$10*labor + \$100*machinery
  - $1000 = 10*x_1 + 100*x_2$
    - Where C = 1000, x_1 = labor, x_2 = machinery

Example of Iso-Cost Line Graphically



Finding the Slope of the Iso-Cost Line

$$1000 = 10x_1 + 100x_2$$

$$\Rightarrow 100x_2 = 1000 - 10x_1$$

$$\Rightarrow \frac{100x_2}{100} = \frac{1000}{100} - \frac{10x_1}{100}$$

$$\Rightarrow x_2 = 10 - \frac{x_1}{10}$$

$$\frac{dx_2}{dx_1} = -\frac{1}{10}$$

Notes on Iso-Cost Line

- As you increase C, you shift the iso-cost line parallel out.
- As you change one of the costs of an input, the iso-cost line rotates.

Cost Minimization with Two Variable Inputs

- Assume that we have two variable inputs (x_1 and x_2) which cost respectively w_1 and w_2. We have a total fixed cost of TFC.
- Assume that the general production function can be represented as $y = f(x_1,x_2)$.

$$\underset{w.r.t.\, x_1, x_2}{Min} \quad w_1 x_1 + w_2 x_2 + TFC$$

$$\text{subject to}: y = f(x_1, x_2)$$

First Order Conditions for the Cost Minimization Problem with Two Inputs

$$\Gamma(x_1, x_2, \lambda) = w_1 x_1 + w_2 x_2 + TFC + \lambda\left(y - f(x_1, x_2)\right)$$

$$\frac{\partial \Gamma}{\partial x_1} = w_1 - \lambda \frac{\partial f}{\partial x_1} = 0$$

$$\Rightarrow w_1 - \lambda MPP_{x_1} = 0$$

$$\frac{\partial \Gamma}{\partial x_2} = w_2 - \lambda \frac{\partial f}{\partial x_2} = 0$$

$$\Rightarrow w_2 - \lambda MPP_{x_2} = 0$$

$$\frac{\partial \Gamma}{\partial \lambda} = y - f(x_1, x_2) = 0$$

Implication of MRTS = Slope of Iso-Cost Line

- Slope of iso-cost line = $-w_1/w_2$, where w_2 is the cost of input 2 and w_1 is cost of input 1.
- MRTS = $-MPP_{x1}/MPP_{x2}$
- This implies $MPP_{x1}/MPP_{x2} = w_1/w_2$
- Which implies $MPP_{x1} / w_1 = MPP_{x2}/w_2$
- This means that the MPP of input 1 per dollar spent on input 1 should equal MPP of input 2 per dollar spent on input 2.

Example 1 of Cost Minimization with Two Variable Inputs

- Suppose you have the following production function:
  - $y = f(x_1,x_2) = 10x_1^{½} x_2^{½}$
- Suppose the price of input 1 is \$1 and the price of input 2 is \$4. Also suppose that TFC = 100.
- What is the optimal amount of input 1 and 2 if you want to produce 20 units.

Example 1 of Cost Minimization with Two Variable Inputs Cont.

- Summary of what is known:
  - w1 = 1, w2 = 4, TFC = 100
  - $y = 10x_1^{½} x_2^{½}$
  - y = 20

$$\underset{w.r.t.\, x_1, x_2}{Min} \quad x_1 + 4x_2 + 100$$

$$\text{subject to}: y = 10x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

Example 1 of Cost Minimization with Two Variable Inputs Cont.

$$\Gamma(x1, x2, \lambda) = 1x_1 + 4x_2 + 100 + \lambda\left(y - 10x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}\right)$$

$$\frac{\partial\Gamma}{\partial x_1} = 1 - \lambda 10\left(\frac{1}{2}\right)x_1^{-\frac{1}{2}}x_2^{\frac{1}{2}} = 0$$

$$\frac{\partial\Gamma}{\partial x_2} = 4 - \lambda 10\left(\frac{1}{2}\right)x_1^{\frac{1}{2}}x_2^{-\frac{1}{2}} = 0$$

$$\frac{\partial\Gamma}{\partial \lambda} = y - 10x_1^{\frac{1}{2}}x_2^{\frac{1}{2}} = 0$$

Solution done in class

Solving Example 1 Using Ratio of MPP's Equals Absolute Value of the Slope of the Cost Function

1. $|MRTS| = \frac{MPP_{x_1}}{MPP_{x_2}} = \frac{x_2}{x_1}$

2. Set $|MRTS| = \frac{w_1}{w_2} = \frac{1}{4}$

$\Rightarrow \frac{x_2}{x_1} = \frac{1}{4}$

$\Rightarrow 4x_2 = x_1$

3. Plug into production function

$y = 10(4x_2)^{\frac{1}{2}}(x_2)^{\frac{1}{2}}$

$\Rightarrow y = 20x_2^{\frac{1}{2}}x_2^{\frac{1}{2}}$

$x_2 = \frac{y}{20}$ *and* $x_1 = \frac{4y}{20}$

$\Rightarrow x_2 = 1$

$\Rightarrow x_1 = 4$

Solving Example 1 Using MRTS from the Isoquant and Setting it Equal to the Slope of the Cost Function

1. Find Isoquant

$$x_2 = \frac{y^2}{100x_1} = \frac{y^2}{100} x_1^{-1}$$

2. Find MRTS

$$\text{MRTS} = \frac{\partial x_2}{\partial x_1} = -\frac{y^2}{100} x_1^{-2}$$

3. Set $MRTS = -\frac{w_1}{w_2} = -\frac{1}{4}$

$$\Rightarrow -\frac{y^2}{100} x_1^{-2} = -\frac{1}{4}$$

$$\Rightarrow x_1 = \frac{4y}{20} = \frac{y}{5}$$

4. Solve for x_2 using x_1

$$x_2 = \frac{y^2}{100}\left(\frac{y}{5}\right)^{-1} = \frac{y}{20}$$

$$\Rightarrow x_2 = 1$$

$$\Rightarrow x_1 = 4$$

Final Note on Input Selection

- You want to have the iso-cost line tangent to the isoquant.
  - This implies that you will set the absolute value of MRTS equal to the absolute value of the slope of the iso-cost line.