Unconstrained and Constrained Optimization

Agenda

- General Ideas of Optimization
- Interpreting the First Derivative
- Interpreting the Second Derivative
- Unconstrained Optimization
- Constrained Optimization

Optimization

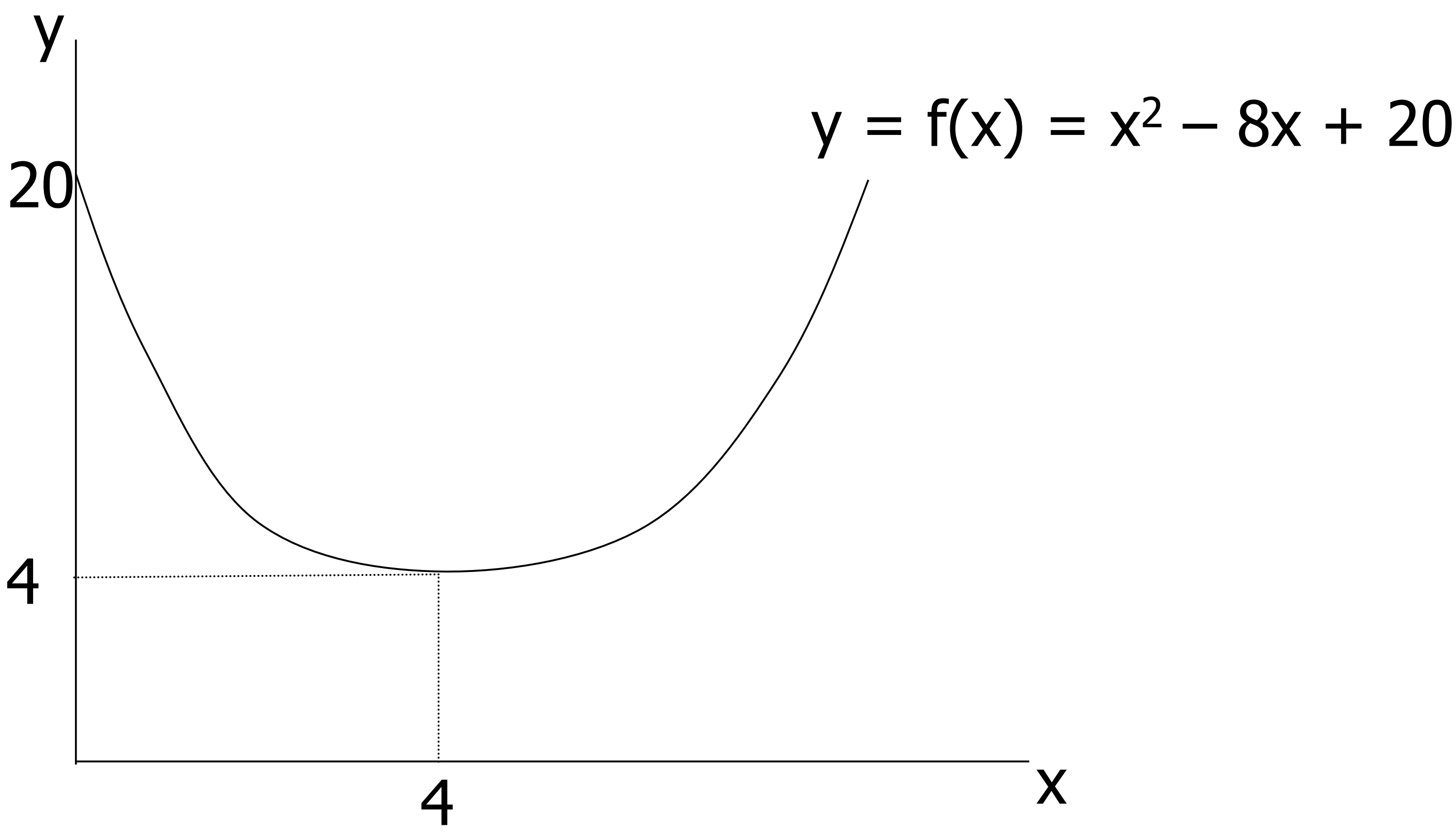
- There are two ways of examining optimization.
  - Minimization
    - In this case you are looking for the lowest point on the function.
  - Maximization
    - In this case you are looking for the highest point on the function.

Needed Terminology Critical Point

- A point x* on a function is said to be a critical point if when you evaluate the derivative of the function at the point x*, then the derivative at that point is zero, i.e., $f'(x^*) = 0$.

What observations can you make about the attributes of a minimum?

Graphical Representation of a Minimum



Questions Regarding the Minimum

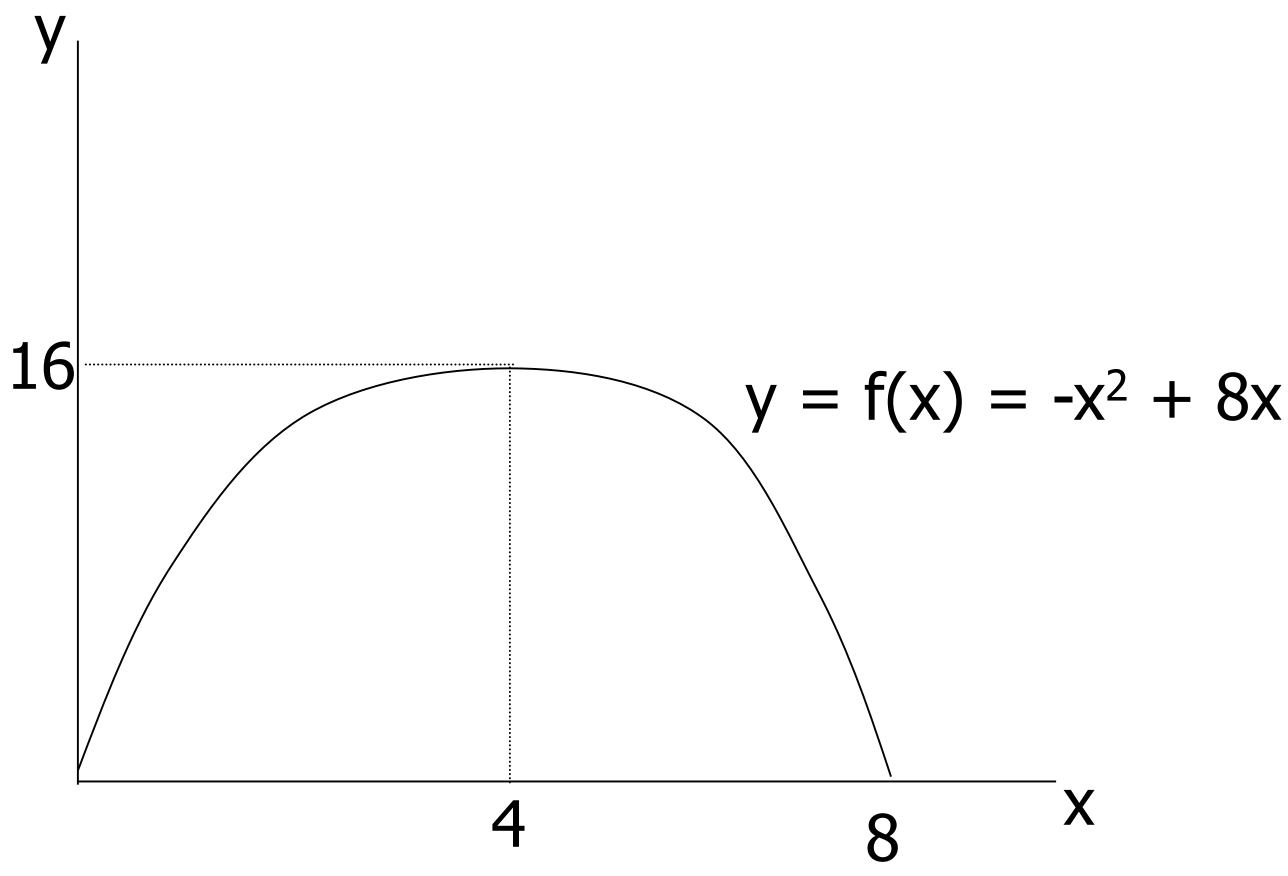
- What is the sign of the slope when you are to the left of the minimum point?
  - Another way of saying this is what is $f'(x)$ when $x < x^*$?
  - Note: x^* denotes the point where the function is at a minimum.

Questions Regarding the Minimum Cont.

- What is the sign of the slope when you are to the right of the minimum point?
  - Another way of saying this is what is $f'(x)$ when $x > x^*$?
- What is the sign of the slope when you at the minimum point?
  - Another way of saying this is what is $f'(x)$ when $x = x^*$?

What observations can you make about the attributes of a maximum?

Graphical Representation of a Maximum



Questions Regarding the Maximum

- What is the sign of the slope when you are to the left of the maximum point?
  - Another way of saying this is what is $f'(x)$ when $x < x^*$?
  - Note: x^* denotes the point where the function is at a maximum.

Questions Regarding the Maximum Cont.

- What is the sign of the slope when you are to the right of the maximum point?
  - Another way of saying this is what is $f'(x)$ when $x > x^*$?
- What is the sign of the slope when you at the maximum point?
  - Another way of saying this is what is $f'(x)$ when $x = x^*$?

Interpreting the First Derivative

- The first derivative of a function as was shown previously is the slope of the curve evaluated at a particular point.
  - In essence it tells you the instantaneous rate of change of the function at the given particular point.
  - Knowing the slope of the function can tell you where a maximum or a minimum exists on a curve.
    - Why?

Question

- Can the derivative tell you whether you are at a maximum or a minimum?
  - The answer is yes if you examine the slope of the function around the critical point, i.e., the point where the derivative is zero.
  - An easier way of examining whether you have a maximum or a minimum is to examine the second derivative of the function.

The Second Derivative

- The second derivative of a function f(x) is the derivative of the function f′(x), where f′(x) is the derivative of f(x).
  - The second derivative can tell you whether the function is concave or convex at the critical point.
  - The second derivative can be denoted by f″(x).

Concavity and the Second Derivative

- The maximum of a function f(x) occurs when a critical point x* is at a concave portion of the function.
  - This is equivalent to saying that $f''(x^*) < 0$.
  - If $f''(x) < 0$ for all x, then the function is said to be concave.

Convexity and the Second Derivative

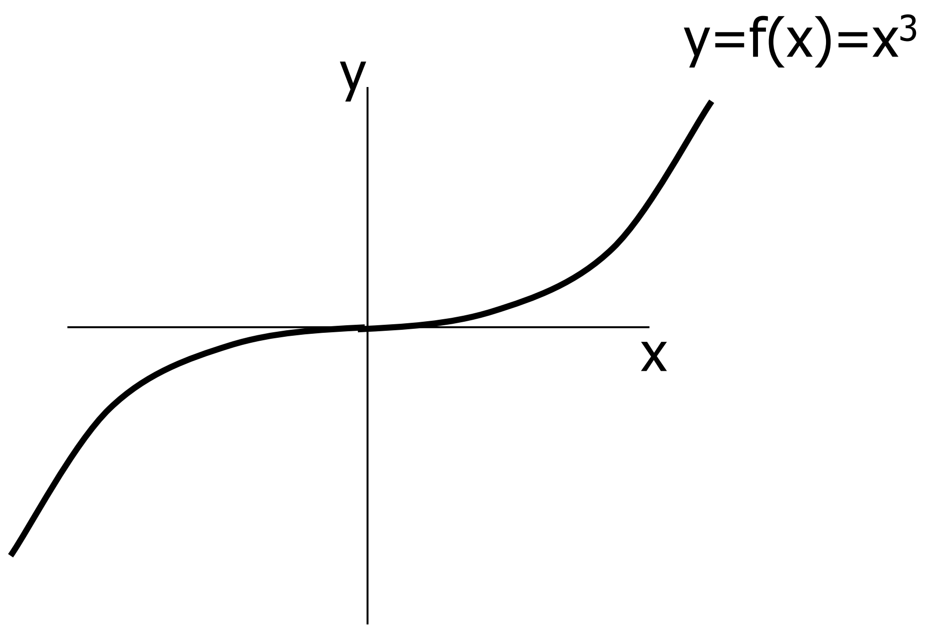
- The minimum of a function f(x) occurs when a critical point x* is at a convex portion of the function.
  - This is equivalent to saying that $f''(x^*) > 0$.
  - If $f''(x) > 0$ for all x, then the function is said to be convex.

Special Case of the Second Derivative

- Suppose you have a function f(x) that has a maximum at x*.
  - What does it mean when the second derivative is equal to zero, i.e., $f''(x^*) = 0$?
  - This is a point where the second derivative may not be able to tell you whether you have a maximum or a minimum.
  - Usually in this case you will get a saddle point/point of inflection where the point is neither a maximum nor a minimum.

Example of Special Case of the Second Derivative

- Suppose $y = f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$,
  - This implies that $x^* = 0$ and $f''(x^*=0) = 0$.



Unconstrained Optimization

- An unconstrained optimization problem is one where you only have to be concerned with the objective function you are trying to optimize.
  - An objective function is a function that you are trying to optimize.
    - None of the variables in the objective function are constrained.

First and Second Order Condition For a Maximum

- The first order condition for a maximum at a point x^* on the function $f(x)$ is when $f'(x^*) = 0$.
- The second order condition for a maximum at a point x^* on the function $f(x)$ is when $f''(x^*) < 0$.

First and Second Order Condition For a Minimum

- The first order condition for a minimum at a point x^* on the function $f(x)$ is when $f'(x^*) = 0$.
- The second order condition for a minimum at a point x^* on the function $f(x)$ is when $f''(x^*) > 0$.

Example of Using First and Second Order Conditions

- Suppose you have the following function:
  - $f(x) = x^3 - 6x^2 + 9x$
- Then the first order condition to find the critical points is:
  - $f'(x) = 3x^2 - 12x + 9 = 0$
  - This implies that the critical points are at $x = 1$ and $x = 3$.

Example of Using First and Second Order Conditions Cont.

- The next step is to determine whether the critical points are maximums or minimums.
  - These can be found by using the second order condition.
  - $f''(x) = 6x - 12 = 6(x-2)$

Example of Using First and Second Order Conditions Cont.

- Testing $x = 1$ implies:
  - $f''(1) = 6(1-2) = -6 < 0$.
  - Hence at $x = 1$, we have a maximum.
- Testing $x = 3$ implies:
  - $f''(3) = 6(3-2) = 6 > 0$.
  - Hence at $x = 3$, we have a minimum.
- Are these the ultimate maximum and minimum of the function $f(x)$?

Relative Vs. Absolute Extremum

- A relative extremum is a point that is locally greater or lesser than all points around it.
  - A relative extrema can be found by using the first order condition.
- An absolute extremum is a point that is either absolutely greater than or less than all other points, i.e., $f(x^*) > f(x)$ for all x not equal to x^* for a maximum and $f(x^*) < f(x)$ for all x not equal to x^* for a minimum.

Finding the Absolute Extremum

- To find the absolute extremum, you need to compare all the critical points on the function, as well as, any potential end points of the function like ∞ and - ∞.
  - When evaluating a polynomial function at ∞, the value of the function at ∞ takes the value of the at the highest ordered variable.

Finding the Absolute Extremum Cont.

- Some properties of ∞:
  - $\infty + \infty = \infty$
  - $\infty - \infty$ is undefined
  - $\infty + c = \infty$, where $-\infty < c < \infty$
  - $c*\infty = \infty$, where c is any value greater than zero
  - $\infty * \infty = \infty$
  - $\infty * (-\infty) = -\infty$
- From the previous example, the relative extremum points occur at $x = -\infty$, 1, 3, and ∞.
- The absolute maximum occurs at $x = \infty$ and the absolute minimum occurs at $x = -\infty$.

Unconstrained Optimization: Two Variables

- Suppose you have a function $y = f(x_1,x_2)$, then to find the critical points, you can use the following first order condition:

$$f_{x_1} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} = 0$$

$$f_{x_2} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} = 0$$

Unconstrained Optimization: Two Variables Cont.

- The second order condition are more complex where you have to examine the second derivative of each of the variables, as well as, the cross derivative.

Cross Partial Derivative

- Suppose you have a function y = f(x_1,x_2), then the cross partial derivative can be represented as:

$$f_{x_i x_j} = \frac{\partial^2 f(x_1, x_2)}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_i \partial x_j}$$

$$f_{x_1 x_2} = \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = \frac{\partial^2 y}{\partial x_1 \partial x_2}$$

$$f_{x_2 x_1} = \frac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} = \frac{\partial^2 y}{\partial x_2 \partial x_1}$$

Cross Partial Derivative Example

- Suppose you have a function $y = f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2$, then:

$$\frac{\partial^2 y}{\partial x_1 \partial x_2} = 2$$

$$\frac{\partial^2 y}{\partial x_2 \partial x_1} = 2$$

Constrained Optimization

- Constrained Optimization is said to occur when one or more of the variables in the objective function is constrained by some function.
  - Hence a constrained optimization problem will have an objective functions and a set of constraints.

Constrained Optimization Cont.

- The constrained optimization problem where you are trying to maximize can be set-up as the following:
  - Maximize an objective function f(x) with respect to x given a set of constraints g(x)=c.

$$\max_{w.r.t.\,x} f(x)$$

$$\text{subject to } g(x) = c$$

Example of Constrained Optimization

- Suppose that you want to maximize $f(x) = 5x - x^2$, subject to the constraint that $x = 2$.
  - Since $x = 2$ is the constraint, the answer to this is trivial where $x^* = 2$.

Constrained Optimization: Two Variables

- The Constrained Optimization problem where you are trying to maximize can be set-up as the following:
  - Maximize an objective function $f(x_1,x_2)$ with respect to x_1,x_2 given a set of constraints $g(x_1,x_2) = c$.

$$\max_{w.r.t.\, x_1, x_2} f(x_1, x_2)$$

$$\text{subject to } g(x_1, x_2) = c$$

Example of Constrained Optimization: Two Variables

- Suppose you want to maximize $y = f(x_1,x_2) = x_1x_2$ with respect to x_1 and x_2 given that $600 = x_1 + 2x_2$.
- To solve this problem, we can turn this constrained problem into an unconstrained problem.

Example of Constrained Optimization: Two Variables Cont.

- If we solve the constraint for x_1 as a function of x_2 we get $x_1 = 600 - 2x_2$.
- Plugging x_1 into the objective function gives the following new unconstrained maximization problem.
- Maximize $(600 - 2x_2)x_2$ w.r.t. x_2.
- The first order condition is $600 - 4x_2=0$.
  - Which implies $x_2^*=150$.
  - Which implies $x_1^*=300$.

Motivating the Lagrange Method

- In the previous problem, we made a substitution to turn the constrained optimization problem into an unconstrained problem.
  - While this made solving the problem easier, there may be times when you have multiple constraints or potentially inequality constraints that make changing the constrained into the unconstrained difficult or impossible.

Motivating the Lagrange Method Cont.

- Another way of solving the above problem is using Lagrange's method.
  - The Lagrange method uses what is called a Lagrange multiplier λ to transform the problem from a constrained problem to an unconstrained problem.

Setting-Up the Lagrange

- The constrained optimization problem where you are trying to maximize/minimize can be set-up as the following:
  - Maximize/minimize an objective function $f(x_1,x_2)$ with respect to x_1 and x_2 given a set of constraints $g(x_1,x_2) = c$.

If a maximum problem

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda(g(x_1, x_2) - c)$$

If a minimum problem

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda(c - g(x_1, x_2))$$

Solving the Lagrange Problem

- To solve the Lagrange problem, you need to optimize $L(x_1,x_2, \lambda)$ with respect to x_1, x_2, and λ.
  - This is equivalent to using the first and second order conditions.

Example of Lagrange

- Suppose you want to maximize $y = f(x_1,x_2) = x_1x_2$ with respect to x_1 and x_2 given that $600 = x_1 + 2x_2$.
  - This implies:

$$L(x_1, x_2, \lambda) = x_1x_2 + \lambda(x_1 + 2x_2 - 600)$$

First Order Conditions

$$\mathrm{L}_{x_1} = x_2 + \lambda = 0$$

$$\mathrm{L}_{x_2} = x_1 + 2\lambda = 0$$

$$\mathrm{L}_{\lambda} = x_1 + 2x_2 - 600 = 0$$